A little more on divergence

Recall the definition of the divergence of a vector field.

Definition. Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

To understand what divergence measures in the case of our velocity field of a planar fluid, we consider a different path integral. Given a simple, closed curve C in the plane, consider the path integral

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds,$$

where $\mathbf{n}$ is a unit normal vector to C that points outside the region enclosed by C.

Theorem. (planar Divergence Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA.$$

This identity justifies the name "divergence."

Surface integrals

A path integral is how we "add up" a function over a curve in the plane or in space. Similarly, a surface integral is how we add up a function $f(x, y, z)$ over a surface in space.

We want an integral such that

$$\iint_S f(x,y,z)\,dS = \text{ the "sum" of } f(x,y,z) \text{ over the surface } S$$
$$= (\text{the average of } f(x,y,z) \text{ over } S)(\text{surface area}(S)).$$

Special case: Assume that the surface S is the graph of a function $g(x, y)$ over a region R in the xy-plane. In this case, the differential of surface area is

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dA.$$

This differential can also be expressed as

$$dS = |\mathbf{N}|\, dA,$$

where the vector

$$\mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface (see the handout for November 6).

Consequently, we have

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dA.$$

Example. Let S be the portion of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. Let's find its center of mass.

If the center of mass is denoted $(\overline{x}, \overline{y}, \overline{z})$, then we know that $\overline{x} = \overline{y} = 0$. Also,

$$\overline{z} = \frac{\displaystyle\iint_S z\, dS}{\text{area}(S)}.$$