

Flux integrals

Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

and a surface S , the flux of \mathbf{F} across S is the surface integral

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS,$$

where \mathbf{n} is the unit normal vector at each point of S .

Special case: Assume that the surface S is the graph of a function $g(x, y)$ over a region R in the xy -plane. In this case, the differential of surface area is

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dA.$$

This differential can also be expressed as

$$dS = |\mathbf{N}| \, dA,$$

where the vector

$$\mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface (see the handout for November 6).

In this case, we compute the flux of \mathbf{F} across S by

$$\begin{aligned}\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \pm \iint_R \left(\mathbf{F} \cdot \frac{\mathbf{N}}{|\mathbf{N}|} \right) |\mathbf{N}| \, dA \\ &= \pm \iint_R (\mathbf{F} \cdot \mathbf{N}) \, dA.\end{aligned}$$

Example. Consider the surface S that is the boundary of the solid that is bounded by the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy -plane. Also, consider the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 3\mathbf{k}.$$

Let's calculate the flux across S in the direction of the outer normal.

The surface S splits nicely into two surfaces. Let's denote the part of S given by the paraboloid as S_1 and the part of S that lies in the xy -plane as S_2 .

Flux across S_1 :

Flux across S_2 :