More on the cross product

Last class we started our discussion of the cross product of two vectors, and we focused on its geometric properties. Today we will derive a formula for the cross product and do some applications.

**Example.** Compute \((4\mathbf{i} + \mathbf{j}) \times (7\mathbf{i} + 2\mathbf{k})\).

To get a general formula, we can repeat the same type of calculation on

\[(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}).\]

The result is

\[(a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k.\]

There is a handy way of remembering this formula that uses the determinant of a \(3 \times 3\) matrix.

We can calculate the determinant

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix}
\]

in one of two ways.

1. **Expansion by minors:** The determinant

\[
\begin{vmatrix}
a & b \\
c & d
\end{vmatrix} = ad - bc,
\]

and the determinant

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix} = \alpha_1 \begin{vmatrix} \beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} - \alpha_2 \begin{vmatrix} \beta_1 & \beta_3 \\ \gamma_1 & \gamma_3 \end{vmatrix} + \alpha_3 \begin{vmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}.
\]

This method is the one that your textbook describes.
2. Sum of six terms, each a product of three numbers: We can also compute this determinant by writing it as

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 \\
\beta_1 & \beta_2 & \beta_3 & \beta_1 & \beta_2 \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_1 & \gamma_2
\end{vmatrix}
\]

Then we get

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix}
= (\alpha_1\beta_2\gamma_3) + (\alpha_2\beta_3\gamma_1) + (\alpha_3\beta_1\gamma_2)
- (\alpha_3\beta_2\gamma_1) - (\alpha_1\beta_3\gamma_2) - (\alpha_2\beta_1\gamma_3)
\]

Using this formula, we can write

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
i & j & k \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}.
\]

**Example.** We use the determinant formula to calculate \((4\mathbf{i} + \mathbf{j}) \times (7\mathbf{i} + 2\mathbf{k})\).
Applications to Lines and Planes

We have discussed finding the equation of a plane if we are given a point \( P \) on the plane and a normal vector \( \mathbf{N} \). Now that we can compute the cross product of two vectors, we are able to find the equation of a plane determined in more familiar ways, e.g., determined by three noncollinear points or by two intersecting lines. We use the cross product to determine \( \mathbf{N} \).

**Example.** Find an equation for the plane that contains the three points \( P_1 = (1, 1, 1) \), \( P_2 = (2, -2, 2) \), and \( P_3 = (0, 2, 1) \).
Example. Find the equation of the plane that contains the two lines

\[ x = y = z \quad \text{and} \quad \frac{x - 1}{2} = \frac{y - 1}{3} = z - 1. \]

Why do these lines intersect? Where do they intersect? What are their direction vectors \( \mathbf{D}_1 \) and \( \mathbf{D}_2 \)?
Example. Let \( \ell_1 \) be line through \((1,0,0)\) and \((1,2,2)\). Let \( \ell_2 \) be the line through \((1,1,1)\) and \((1 + \sqrt{2}, 2, 2)\). Do these lines intersect? What is the angle of intersection?