

## Vector-valued functions and parameterized curves

First, let's quickly review parameterized curves in the plane (see Section 1.7 of your text).

**Definition.** A *parametric curve* in the  $xy$ -plane is a pair of scalar functions

$$\begin{aligned}x &= f(t) \\ y &= g(t)\end{aligned}$$

We trace out the curve by plotting all points of the form

$$\text{trace} = \{(f(t), g(t)) \mid \text{for all } t \text{ in the domains of } f \text{ and } g\}.$$

**Example.**

$$\begin{aligned}x(t) &= 3t + 1 \\ y(t) &= 2t + 2\end{aligned}$$

| t  | x  | y |
|----|----|---|
| -1 | -2 | 0 |
| 0  | 1  | 2 |
| 1  | 4  | 4 |
| 2  | 7  | 6 |

We can solve for  $t$  to get a nonparametric representation of the curve.

$$x = 3t + 1$$

$$x - 1 = 3t$$

$$\frac{x - 1}{3} = t$$

Therefore, we have

$$y = 2 \left( \frac{x - 1}{3} \right) + 2$$

$$= \frac{2}{3}x - \frac{2}{3} + 2$$

$$= \frac{2}{3}x + \frac{4}{3}.$$

**Remark.** Any parameterized equation of the form

$$\begin{aligned}x &= at + b \\ y &= ct + d\end{aligned}$$

is a line.

In this course, you need to know how to parameterize any line in the plane. For practice, start with two points  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$  and form parametric equations for the line that contains  $P_1$  and  $P_2$ .

Curves in space can be described in essentially the same manner as curves in the plane. Their parametric representation is given by three scalar-valued functions

$$\begin{aligned}x &= f(t) \\ y &= g(t) \\ z &= h(t).\end{aligned}$$

### Vector-valued functions

When we study curves in the plane or in space, it is often useful to employ vector techniques, and we do so by using vector-valued functions.

Given a parameterized curve in space of the form

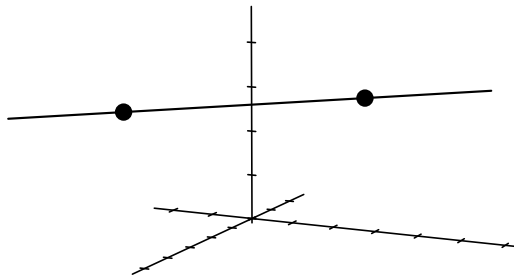
$$\begin{aligned}x &= f(t) \\ y &= g(t) \\ z &= h(t),\end{aligned}$$

we can combine these three functions to make one vector-valued function

$$\mathbf{P}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

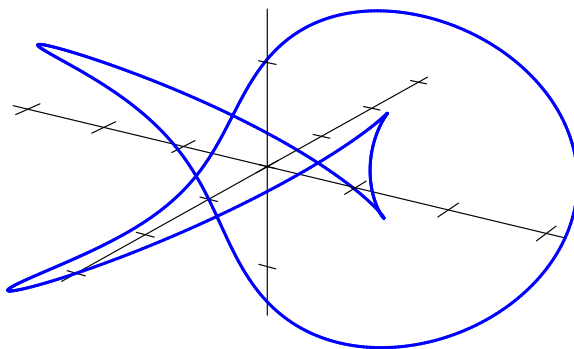
The vector  $\mathbf{P}(t)$  is often thought of as a position vector that varies with the parameter  $t$ .

**Example 1.** Let  $\mathbf{L}(t) = (4 - t)\mathbf{i} + (5t - 1)\mathbf{j} + (3 + \frac{1}{2}t)\mathbf{k}$ .



**Example 2.** Let  $\mathbf{H}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ .

**Example 3.** Let  $\mathbf{K}(t) = \left((2 + \cos \frac{3}{2}t) \cos t\right)\mathbf{i} + \left((2 + \cos \frac{3}{2}t) \sin t\right)\mathbf{j} + \left(\sin \frac{3}{2}t\right)\mathbf{k}$ .



In one-dimensional calculus, we define the derivative of a scalar-valued function as the limit

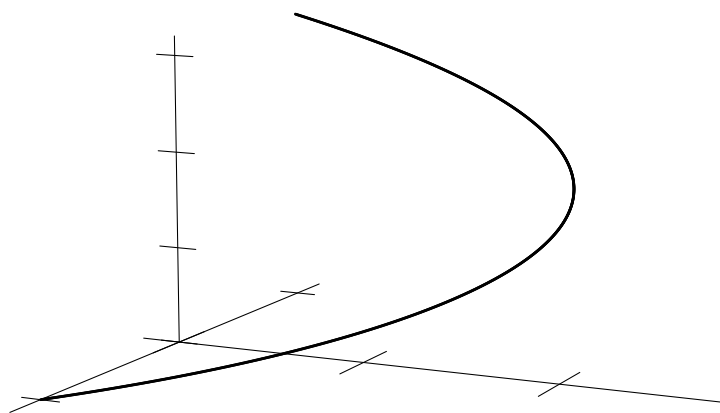
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

It is the limit of the change in  $f(x)$  divided by the change in  $x$ . We can do the same for vector-valued functions.

**Definition.** Let  $\mathbf{f}(t)$  be a vector-valued function. Then

$$\mathbf{f}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

What does this limit represent? First, let's consider the definition in terms of motion in space.



We see that the secants limit on a tangent vector. We divide by  $h$  to stop the vectors from shrinking to zero. As we shall see on Wednesday, there is another good reason for dividing by  $h$ .

Since we have this interesting vector associated to  $\mathbf{f}(t)$ , how do we compute it?

**Theorem.** Let  $\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Then  $\mathbf{f}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ .

**Example.** Consider a curve that is very similar to the circular helix. Let

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t\mathbf{k}.$$

Then

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + \mathbf{k}.$$

So the derivative at  $t = \pi/3$  is

$$-\frac{\sqrt{3}}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

This vector is tangent to the elliptical helix at  $t = \pi/3$ .

**Example.** Find the equation of the tangent line to the curve

$$\mathbf{r}(t) = e^t \mathbf{i} + 2 \sin t \mathbf{j} + (t^2 - 2) \mathbf{k}.$$

at the point  $(1, 0, -2)$ .