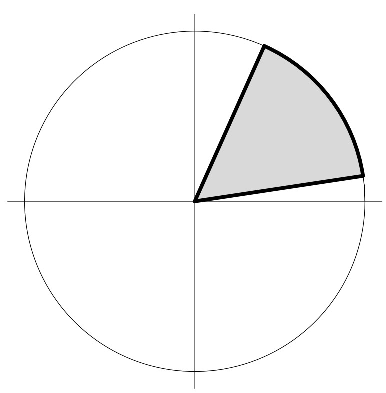
Area of regions enclosed by polar curves

Last class we talked about curves in the plane determined by polar equations. In particular, most of the examples that we discussed were of the form $r = f(\theta)$. Today we will study how to calculate the area of regions determined by such curves.

Here is the abstract picture:

In order to derive the general formula, we need a basic fact about the area of a circular sector.



To derive the area formula, we perform a typical Riemann sums argument applied to the θ-interval $\alpha \leq \theta \leq \beta$. Partition the polar region into n equal θ-width subsectors

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_n = \beta,$$

where the width is $\Delta\theta = (\beta - \alpha)/n$ and $\theta_k = \theta_{k-1} + \Delta\theta$.

In each "subsector" corresponding to $\theta_k \leq \theta \leq \theta_{k-1}$, pick an angle t_k. We can use the area formula for a sector of a circle derived above to approximate the area of the subsector.

Area formula. Area $A = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$

Example 1. Find the area of the region enclosed by one loop of the polar curve $r = 2\sin 3\theta$.

Example 2. Find the area of the region inside one loop of the curve $r = 2\sin 3\theta$ but outside the circle $r = 1$.

Differential Notation. A *differential* is an expression that you integrate to get the quantity that you want. In this course, we will use the variable A to represent area in the plane, and the differential version of the polar area formula is

$$dA = \tfrac{1}{2} r^2 \, d\theta.$$

Another example of differential notation is the formula for arc length that we learned earlier. We will often use the variable s to denote arc length. You integrate the differential ds to calculate arc length. You may see the formula

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2},$$

where the equations for (x, y, z) determine the curve. In other words,

$$\begin{aligned} s &= \int ds \\ &= \int \left(\frac{ds}{dt}\right) dt \\ &= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt, \end{aligned}$$

which agrees with our arc length formula

$$\text{arc length } = \int |\mathbf{f}'(t)| \, dt$$

because

$$\mathbf{f}'(t) = \left(\frac{dx}{dt}\right)\mathbf{i} + \left(\frac{dy}{dt}\right)\mathbf{j} + \left(\frac{dz}{dt}\right)\mathbf{k}.$$

In the special case of a curve (x, y) in the plane, we have $dz = 0$ and

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

If $y = f(x)$, the differential formula further simplifies to

$$ds = \sqrt{1 + (dy)^2},$$

and if $x = g(y)$, we have

$$ds = \sqrt{(dx)^2 + 1}.$$

We will sometimes use differential notation when it is convenient. At first, it is somewhat confusing, but once you get used to it, it is very handy.