1. (21 points) Let $f(x, y) = e^{2xy} \sin 3x$. Calculate:

(a) the partial derivative $\dfrac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} = e^{2xy}(\cos 3x)3 + e^{2xy}(2y)(\sin 3x)$$
$$= e^{2xy}(3\cos 3x + 2y \sin 3x)$$

(b) the gradient $\nabla f\left(\dfrac{\pi}{18}, 0\right)$

$$\frac{\partial f}{\partial y} = 2x e^{2xy} \sin 3x$$
$$\frac{\partial f}{\partial x}\left(\frac{\pi}{18}, 0\right) = 3\cos\frac{\pi}{6} = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$
$$\frac{\partial f}{\partial y}\left(\frac{\pi}{18}, 0\right) = 2\left(\frac{\pi}{18}\right)\sin\frac{\pi}{6} = 2\left(\frac{\pi}{18}\right)\left(\frac{1}{2}\right) = \frac{\pi}{18}$$
$$\nabla f\left(\frac{\pi}{18}, 0\right) = \frac{3\sqrt{3}}{2}\vec{i} + \frac{\pi}{18}\vec{j}$$

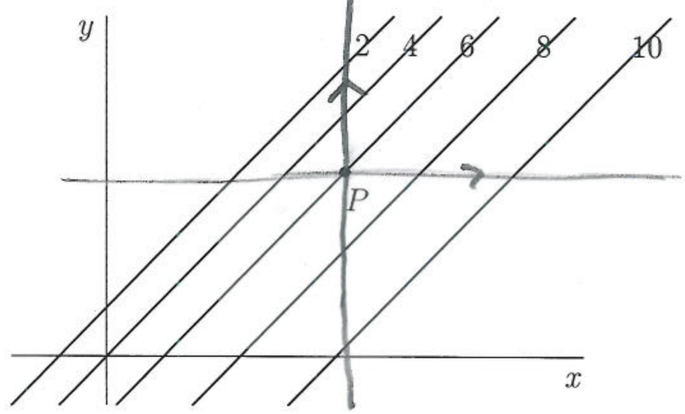
(c) A vector $\mathbf{T}$ that is tangent to the level set of f at the point $\left(\dfrac{\pi}{18}, 0\right)$.

$\nabla f\left(\frac{\pi}{18}, 0\right)$ is perpendicular to the level set, so we need a vector $\vec{T}$ that is perpendicular to ∇f.

$$\vec{T} = \frac{\pi}{18}\vec{i} - \frac{3\sqrt{3}}{2}\vec{j}$$

2. (18 points) The following curves are level sets of a function $f(x, y)$ corresponding to levels 2, 4, 6, 8, and 10.

for $\partial f/\partial y$



for $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$

For each derivative at the point P shown, indicate if it is most likely positive or negative, and provide a brief justification for your answer. **You will not receive any credit unless you provide a valid justification.**

(a) Is $\dfrac{\partial f}{\partial x}$ positive or negative at P? Why?

$\frac{\partial f}{\partial x}(P)$ is probably positive because $f(x,y)$ is increasing in the horizontal direction at P.

(b) Is $\dfrac{\partial f}{\partial y}$ positive or negative at P? Why?

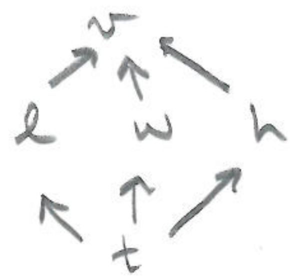
$\frac{\partial f}{\partial y}(P)$ is probably negative because $f(x,y)$ is decreasing in the vertical direction at P.

(c) Is $\dfrac{\partial^2 f}{\partial x^2}$ positive or negative at P? Why?

$\frac{\partial^2 f}{\partial x^2}(P)$ is probably negative because the rate at which $f(x,y)$ is increasing is decreasing in the horizontal direction.

3. (18 points) The length l, width w, and height h of a rectangular box are changing with time t.

   (a) State the version of the Chain Rule that specifies how the volume v of the box changes with time.



$$\frac{dv}{dt} = \left(\frac{\partial v}{\partial l}\right)\left(\frac{dl}{dt}\right) + \left(\frac{\partial v}{\partial w}\right)\left(\frac{dw}{dt}\right) + \left(\frac{\partial v}{\partial h}\right)\left(\frac{dh}{dt}\right)$$

   (b) At a given instant t_0, $l = 2$ m, $w = 1$ m, and $h = 3$ m. Also, at the same instant, l is increasing at a rate of 2 m/s, w is increasing at a rate of 3 m/s, and h is decreasing at a rate of 2 m/s. At what rate is the volume of the box changing at time t_0?

$$v = lwh \Rightarrow \frac{\partial v}{\partial l} = wh \qquad \frac{\partial v}{\partial w} = lh$$
$$\frac{\partial v}{\partial h} = lw$$

So at $t = t_0$,
$$\frac{\partial v}{\partial l} = (1)(3) = 3$$
$$\frac{\partial v}{\partial w} = (2)(3) = 6$$
$$\frac{\partial v}{\partial h} = (2)(1) = 2$$

At $t = t_0$,
$$\frac{dv}{dt} = (3)(2) + (6)(3) + (2)(-2)$$
$$= 20 \text{ m}^3/\text{s}.$$

4. (20 points) Find the average value of the function $f(x, y) = 4xye^{x^2y}$ over the rectangle

$$R = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}.$$

First we compute $\iint_R f(x,y)\,dA$.

Use y-slices:

$$\iint_R 4xy e^{x^2y}\,dA = \int_0^3 \int_0^2 4xye^{x^2y}\,dx\,dy$$

$$= \int_0^3 \left[2e^{x^2y}\right]_{x=0}^{x=2} dy$$

$$= 2\int_0^3 (e^{4y} - 1)\,dy$$

$$= 2\left[\frac{e^{4y}}{4} - y\right]_{y=0}^{y=3}$$

$$= 2\left[\frac{e^{12}}{4} - 3 - \left(\frac{1}{4}\right)\right]$$

$$= 2\left[\frac{e^{12}}{4} - \frac{13}{4}\right] = \frac{e^{12}-13}{2}$$

Area of $R = (2)(3) = 6$

$$\text{Average value} = \frac{\iint_R f(x,y)\,dA}{\text{area } R} = \frac{e^{12}-13}{12}$$

5. (20 points) Find the point(s) on the surface $x^2 + y^2 - z^2 = 2$ that are closest to the point $(2, 3, 0)$.

min distance of (x,y,z) to $(2,3,0)$ subject to the constraint $x^2+y^2-z^2=2$.

Use $s = (x-2)^2 + (y-3)^2 + z^2$ (square of distance) subject to constraint $c = x^2+y^2-z^2 = 2$.

$$\nabla s = 2(x-2)\vec{i} + 2(y-3)^2\vec{j} + 2z\vec{k}$$

$$\nabla c = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$$

$$\nabla s = \lambda \nabla c \Rightarrow \cancel{2}(x-2) = \cancel{2}\lambda x$$
$$\cancel{2}(y-3) = \cancel{2}\lambda y$$
$$\cancel{2}z = -\cancel{2}\lambda z$$

The third equation $\Rightarrow$ either $z=0$ or $\lambda = -1$.

If $z=0$, consider the first two equations. multiply the first by y and the second by x to get

$$(x-2)y = \lambda xy = x(y-3).$$

$$\Rightarrow \quad -2y = -3x \quad \Rightarrow \quad y = \tfrac{3}{2}x.$$

$$x^2+y^2-z^2 = 2 \Rightarrow x^2 + \tfrac{9}{4}x^2 = 2$$

$$\tfrac{13}{4}x^2 = 2$$

$$x^2 = \tfrac{8}{13} \Rightarrow x = \pm\tfrac{2\sqrt{2}}{\sqrt{13}}$$

two points: $\left(\frac{2\sqrt{2}}{\sqrt{13}}, \frac{3\sqrt{2}}{\sqrt{13}}, 0\right)$ $\left(-\frac{2\sqrt{2}}{\sqrt{13}}, -\frac{3\sqrt{2}}{\sqrt{13}}, 0\right)$

The first point is definitely closer to $(2,3,0)$ than the second.

If $\lambda = -1$, we get $x-2=-x$
$y-3=-y$

$\Rightarrow$ $x=1$ and $y=3/2$.

Then $z^2 = x^2+y^2-2 = 1+\frac{9}{4}-2 = \frac{5}{4}$

$\Rightarrow z = \pm\frac{\sqrt{5}}{2}$.

two more points: $\left(1, \frac{3}{2}, \pm\frac{\sqrt{5}}{2}\right)$

both points are the same distance from $(2,3,0)$

Need to compare distances for $\left(\frac{2\sqrt{2}}{\sqrt{13}}, \frac{3\sqrt{2}}{\sqrt{13}}, 0\right)$ and $\left(1, \frac{3}{2}, \pm\frac{\sqrt{5}}{2}\right)$

$(\text{Distance})^2$ between $\left(\frac{2\sqrt{2}}{\sqrt{13}}, \frac{3\sqrt{2}}{\sqrt{13}}, 0\right)$ and $(2,3,0)$ is $15-2\sqrt{26} \approx 4.8$

$(\text{Distance})^2$ between $\left(1, \frac{3}{2}, \pm\frac{\sqrt{5}}{2}\right)$ and $(2,3,0)$ is $9/2 = 4.5$.

So the two points $\left(1, \frac{3}{2}, \pm\frac{\sqrt{5}}{2}\right)$ are closest to $(2,3,0)$ on the surface.