Comments on the Fence Problem on Exam 3 (white)

There were two surface area problems on the third exam. The one involving a circular paraboloid was designed to test your understanding of equation 6 on p. 870 and conversion of double integrals to polar coordinates. The one involving the "surface area" of a fence was designed to test your understanding of path integrals.

The fence problem was based on the first example of a path integral that I did in class (see p. 2 of the notetaker helper for November 19). You can also read about this interpretation of path integrals in your textbook on p. 913. Note Figure 2 on that page and the paragraph directly above Example 1.

Unfortunately, many of you wanted to do the fence problem using the formula on p. 870. This method of calculation is much more difficult because the surface is not given as the graph of a function $f(x, y)$ of two variables over a region R in the xy-plane. Typical wrong answers involved the integrals

$$\iint_R \sqrt{1 + (\tfrac{1}{10})^2 + (\tfrac{1}{5})^2}\, dA$$

where the height function was assumed to be $f(x, y)$ and

$$\iint_R \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

where the height function was assumed to be $x^2 + y^2$. In both cases, it is unclear what R should be. Many of you picked the half disk

$$x^2 + y^2 \leq 100 \quad \text{with} \quad y \geq 0.$$

It is possible to do this problem using a variation of the formula on p. 870. The fence is the graph of the function $f(x, z) = \sqrt{100 - x^2}$ over the region

$$R = \left\{ (x, z) \mid -10 \leq x \leq 10,\ 0 \leq z \leq 4 + \frac{x}{5} + \frac{\sqrt{100 - x^2}}{10} \right\}$$

in the xz-plane. In this case,

$$dS = \frac{10}{\sqrt{100 - x^2}}\, dA$$

where A represents area in the xz-plane. The desired surface area is

$$\begin{aligned}
\iint_R dS &= \int_{-10}^{10} \int_0^{4+x/5+\sqrt{100-x^2}/10} \frac{10}{\sqrt{100 - x^2}}\, dz\, dx \\
&= \int_{-10}^{10} \frac{40}{\sqrt{100 - x^2}} + \frac{2x}{\sqrt{100 - x^2}} + 1\, dx \\
&= (40\pi + 20)\ \text{ft}^2.
\end{aligned}$$