Partial derivatives and tangent planes

The partial derivative with respect to y is defined just like the partial derivative with respect to x.

Definition. The partial derivative of $f(x, y)$ in the y-direction at the point (a, b) is defined by

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

We keep x constant and vary y as we take the limit.

Example. Consider $g(x, y) = y \ln(xy) + y$ again and calculate $\partial g/\partial y$ this time.

Example. Consider the function $f(x, y) = 9 - x^2 - y^2$ at the point $(1, 2)$. In what direction, the x-direction or the y-direction, does $f(x, y)$ decrease most rapidly?

Geometric interpretation of partial derivatives

Now let's discuss the geometric significance of the two numbers that we obtain from the partials of $f(x, y) = 9 - x^2 - y^2$ at $(1, 2)$. For example, we can use these numbers to calculate the equation of the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 2, 4)$.

Definition. Suppose that the partial derivatives

$$\frac{\partial f}{\partial x}(a,b) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(a,b)$$

exist at the point (a,b). Then let

$$\mathbf{T}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}(a,b)\right)\mathbf{k} \qquad \text{and} \qquad \mathbf{T}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}(a,b)\right)\mathbf{k}.$$

The normal vector for $f(x,y)$ at the point (a,b) is $\mathbf{N} = \mathbf{T}_y \times \mathbf{T}_x$.

The equation for the tangent plane can be written as

$$z - c = \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b),$$

where $c = f(a,b)$.

Linear approximation

The equation for the tangent plane can also be thought of as a linear approximation to $f(x, y)$ for (x, y) near (a, b).

We can use the the formula for the tangent plane to define a "linear" function

$$L(x, y) = f(a, b) + \left(\frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(\frac{\partial f}{\partial y}(a, b)\right)(y - b).$$

The graph of this function is the tangent plane for $f(x, y)$ at the point (a, b), and it provides a linear approximation to $f(x, y)$ near (a, b).

Example. The linear approximation of the function $f(x, y) = 9 - x^2 - y^2$ near the point $(1, 2)$ is

$$f(x, y) \approx L(x, y) = 4 - 2(x - 1) - 4(y - 2).$$

Another way to write this approximation is as

$$f(1 + \Delta x, 2 + \Delta y) \approx 4 - 2\,\Delta x - 4\,\Delta y,$$

where $\Delta x = x - 1$ and $\Delta y = y - 2$.

Example. Calculate the linear approximation of the function $g(x, y) = y \ln(xy) + y$ near the point $(1/2, 2)$.

Second partials

Just as there is a second derivative for a function of one variable, there are four second partial derivatives for a function of two variables.

Example. Consider $g(x, y) = y \ln(xy) + y$ as discussed earlier. We have already calculated that

$$\frac{\partial g}{\partial x} = \frac{y}{x} \qquad \text{and} \qquad \frac{\partial g}{\partial y} = 2 + \ln(xy).$$

Consequently,

$$\frac{\partial^2 g}{\partial x^2} = -\frac{y}{x^2} \qquad \text{and} \qquad \frac{\partial^2 g}{\partial y^2} = \frac{1}{y}.$$

What about the other two partials

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)?$$

Clairaut's Theorem. If $f(x, y)$ and its partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

are continuous, then the order of partial differentiation is irrelevant. In other words,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

There is a link on the course web page to a discussion of an example for which the conclusion of Clairaut's Theorem does not hold. We will do our best to avoid such functions in this course.