Multivariable chain rules

Today we start our discussion of multivariable chain rules. There are two basic types, and both generalize the single variable chain rule.

Chain Rule—Type I

Consider a vector-valued function \( \mathbf{P}(t) = x(t)i + y(t)j \) which parameterizes a curve in the \( xy \)-plane and a function \( f(x, y) \). How is the derivative \( df/dt \) related to the partial derivatives of \( f(x, y) \) and the derivative \( \mathbf{P}'(t) \)?

**Example.** Consider \( \mathbf{P}(t) = ti + t^2j \) and \( f(x, y) = 2x^2 + y^2 \). Let’s calculate the derivative \( df/dt \) at \( t = 1 \).
The problem with this approach is that it ignores the fact that the function in question is really a composition of two functions. We can illustrate this composition symbolically by making a dependency chart.

**Chain Rule.** The derivative of the composition $f(P(t))$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

**Example.** Back to the composition of $P(t) = ti + t^2j$ and $f(x, y) = 2x^2 + y^2$. Let’s calculate the derivative $df/dt$ at $t = 1$ using the Chain Rule.
This version of the Chain Rule has an important formulation in terms of the gradient of $f$.

**Definition.** Given a function $f(x, y)$ that is differentiable at the point $(a, b)$. Then the *gradient vector* of $f$ at $(a, b)$ is the vector
\[ \nabla f(a, b) = \frac{\partial f}{\partial x}(a, b) \mathbf{i} + \frac{\partial f}{\partial y}(a, b) \mathbf{j}. \]

Sometimes the gradient vector of $f$ is denoted $\text{grad} f(a, b)$.

**Restatement of the Chain Rule.** The derivative of the composition $f(P(t))$ is
\[ \left. \frac{df}{dt} \right|_{t=t_0} = \nabla f(P(t_0)) \cdot P'(t_0). \]

**Example.** Once again we return to $P(t) = ti + t^2j$ and $f(x, y) = 2x^2 + y^2$. 
Animation of this chain rule

**Example.** Use the polar curve $r = \cos 2\theta$ to parameterize a curve $P(t)$ in the $xy$-plane and consider the composition $f(P(t))$ where

$$f(x, y) = y^2 - x^2.$$
This chain rule has some important theoretical implications as well.

**Theorem.**

1. Let $f(x, y)$ be a differentiable function such that $\nabla f(x, y) = 0$ for all $(x, y)$. Then $f(x, y)$ is a constant function.

2. If $g(x, y)$ and $h(x, y)$ are two differentiable functions such that

$$\nabla g(x, y) = \nabla h(x, y)$$

for all $(x, y)$. Then $g(x, y) = h(x, y) + K$ for some constant $K$. 