More on the gradient vector

Last class we saw that we calculate directional derivatives using the Chain Rule.

Theorem. $D_{\mathbf{u}}f(a,b) = [\nabla f(a,b)] \cdot \mathbf{u}$.

This theorem tells us know how a function changes in any given direction. Another useful way to express this result is

$$D_{\mathbf{u}}f(a,b) = |\nabla f(a,b)| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and the gradient vector $\nabla f(a,b)$.

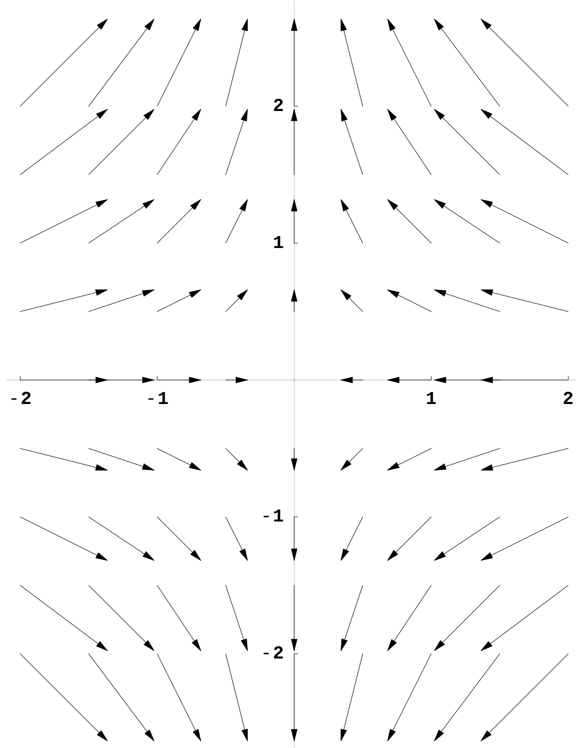
Theorem. The function $f(x,y)$ increases most rapidly in the direction of the gradient. The function is "constant" in directions perpendicular to the gradient.

The animation on the web site by Professor John Putz of Alma College does a good job of illustrating this important fact.

Example. The gradient of $f(x,y) = \frac{1}{4}(y^2 - x^2)$ is

$$\nabla f(x,y) = \tfrac{1}{2}(-x\mathbf{i} + y\mathbf{j}).$$

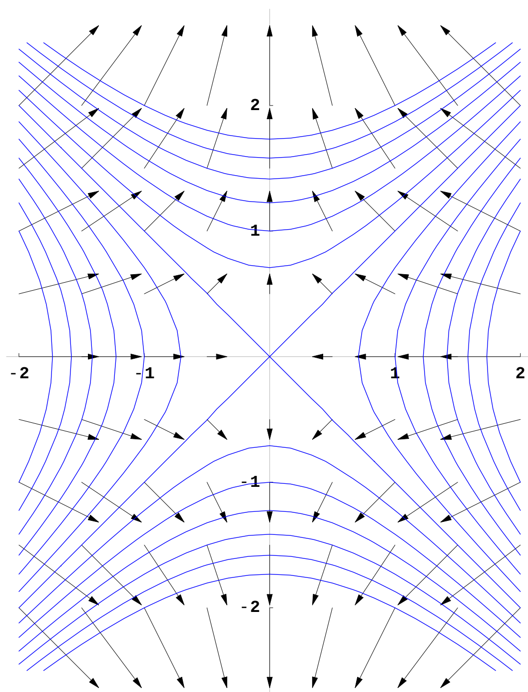
Here is the gradient *vector field* that it generates.



Another important theoretical application of the Chain Rule is the fact that the gradient vector is always perpendicular to its corresponding level set.

Theorem. The gradient vector $\nabla f(a,b)$ is perpendicular to the level set of level $f(a,b)$.

Example. Once again consider $f(x,y) = \frac{1}{4}(y^2 - x^2)$. Its level sets are hyperbolas that are perpendicular to the gradient vector field.



The same theorem holds for functions of any number of variables.

Example. Find an equation for the plane that is tangent to the surface

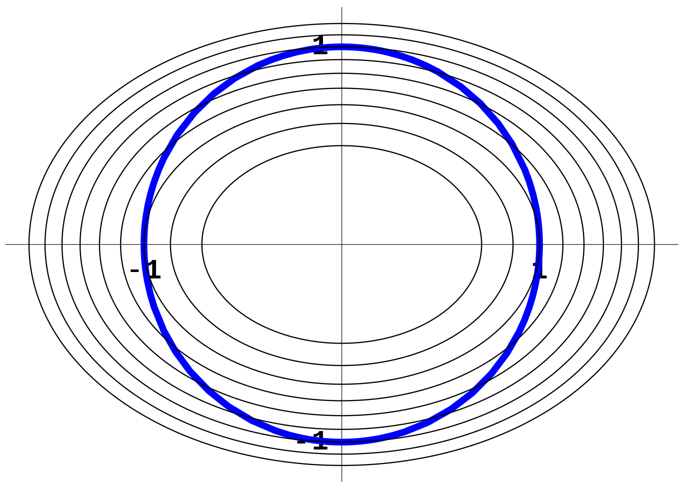
$$x^2 - y^2 + z^2 = 4$$

at the point $(2, -3, 3)$.

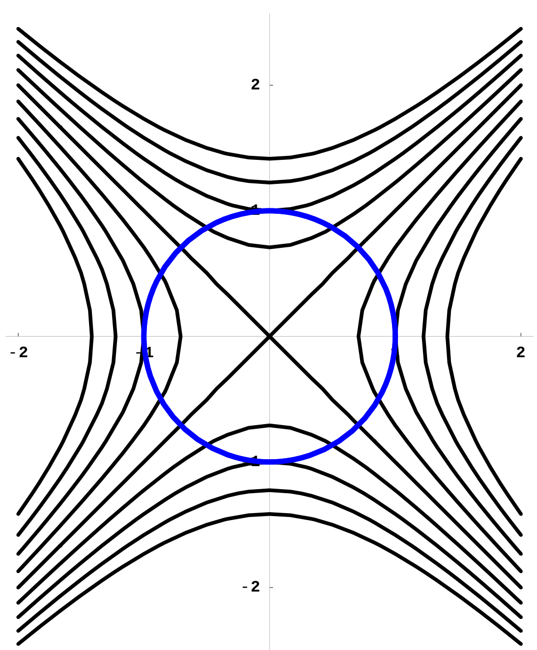
Constrained max/min and the method of Lagrange multipliers

A nice application of the geometry of the gradient is the method of Lagrange multipliers. It is a method that locates extreme values *subject to a constraint*.

Example 1. Consider the function $f(x, y) = 2x^2 + 4y^2$. What are its extreme values if x and y are subject to the constraint $x^2 + y^2 = 1$?



Example 2. A somewhat easier example to analyze is the function $f(x, y) = \frac{1}{4}(y^2 - x^2)$ subject to the same constraint $x^2 + y^2 = 1$. Here are its level sets along with the constraint.



The method of Lagrange multipliers is based on the following theorem.

Theorem. The gradient ∇f is perpendicular to the constraint at the constrained max/min of f.

The method of Lagrange multipliers

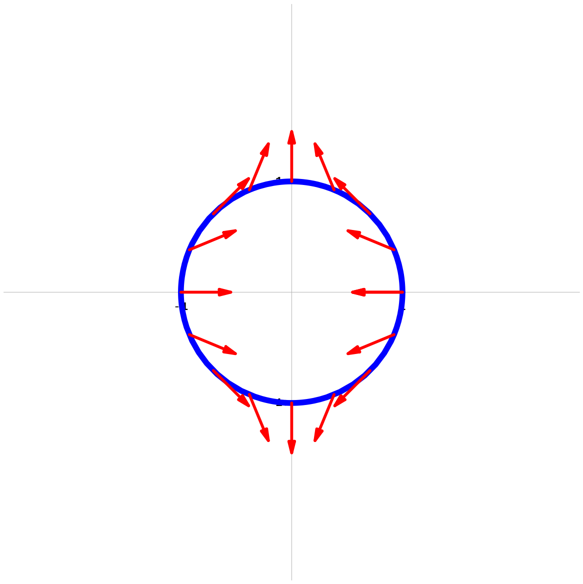
The constraint is also a level set. That is, it is a level set of the constraint function C. If we combine the result of the theorem along with the fact that the gradient of C is perpendicular to the level sets of C, we get the Lagrange multiplier equation

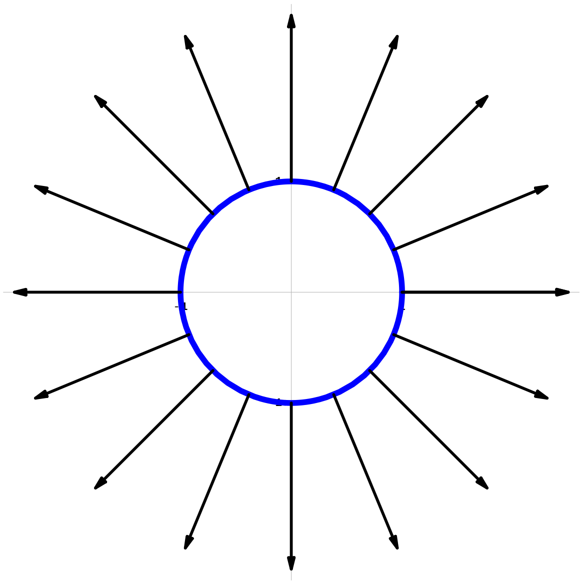
$$\nabla f(P) = \lambda \nabla C(P)$$

for some scalar λ at points P where the constrained max or min occurs.

Example. Back to Example 2: What points (x, y) does the method of Lagrange multipliers identify?

Here are the gradient vectors of both $f(x,y) = \frac{1}{4}(y^2 - x^2)$ and $C(x,y) = x^2 + y^2$ along the constraint $x^2 + y^2 = 1$. The left-hand figure includes the gradient of $f(x,y)$, and the right-hand figure has the gradient of $C(x,y)$.





Here are both gradients in the same figure: