More on the method of Lagrange multipliers

We want to maximize a function $f(x, y)$ subject to a constraint, and the constraint is given as a level set

$$C(x, y) = K$$

of the constraint function $C(x, y)$. Last class we saw that

$$\nabla f(P) = \lambda \nabla C(P)$$

for some scalar λ at points P where the constrained max or min occurs.

Example. Back to Example 2 from last class: Let's find the extreme values of

$$f(x, y) = \tfrac{1}{4}(y^2 - x^2)$$

subject to the constraint $x^2 + y^2 = 1$.

Last class we calculated

$$\nabla f(x, y) = \left(-\frac{x}{2}\right)\mathbf{i} + \left(\frac{y}{2}\right)\mathbf{j}.$$

Also, $C(x, y) = x^2 + y^2$, so

$$\nabla C(x, y) = (2x)\mathbf{i} + (2y)\mathbf{j}.$$

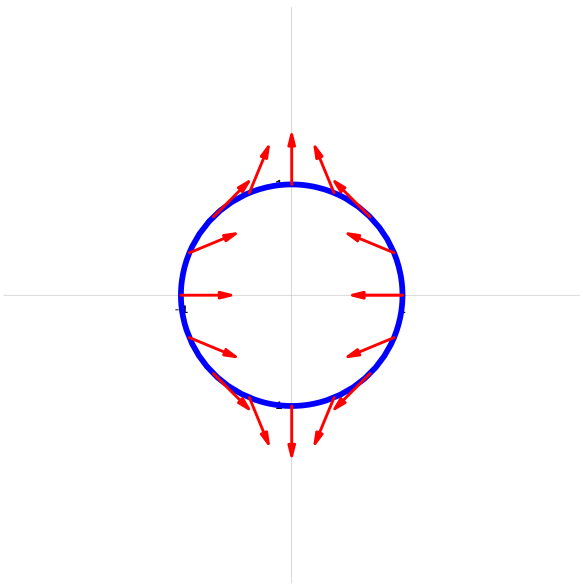
The vector equation

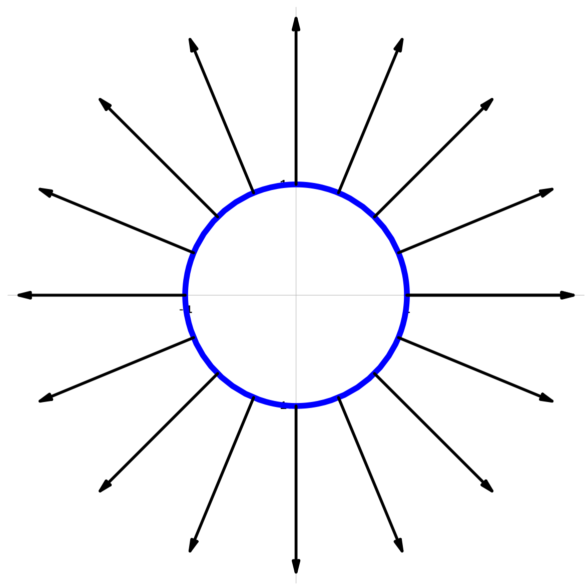
$$\nabla f(x, y) = \lambda \nabla C(x, y)$$

yields the two scalar equations

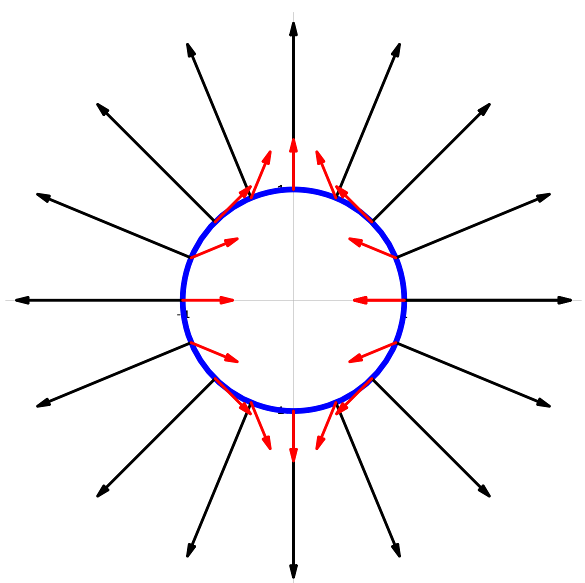
$$\begin{cases} -x = 4\lambda x \\ y = 4\lambda y. \end{cases}$$

Here are the gradient vectors of both $f(x, y) = \frac{1}{4}(y^2 - x^2)$ and $C(x, y) = x^2 + y^2$ along the constraint $x^2 + y^2 = 1$. The left-hand figure includes the gradient of $f(x, y)$, and the right-hand figure has the gradient of $C(x, y)$.





Here are both gradients in the same figure:



Example. Find the point on the plane $x + y + 2z = 1$ that is closest to the origin.

Multiple integrals

We start our discussion of integration by defining the integral of a function $f(x, y)$ of two variables. However, before we get into the details, it helps if we review what we know about the integral

$$\int_a^b f(x)\,dx$$

of a function of one variable. There are three interpretations of $\int_a^b f(x)\,dx$ that I would like you to keep in mind.

You can review these interpretations of the integral on pp. 354–356 and pp. 467–468 of the textbook.

These three interpretations have analogues for the integral

$$\iint_R f(x,y)\,dA$$

of a function of two variables.

The definition of the double integral is also analogous to the definition in the one-variable case. For $\int_a^b f(x)\,dx$, we partition the interval $[a,b]$ into subintervals (usually of equal width).

In each subinterval $[x_{i-1}, x_i]$, we pick a "test" number x_i^*, and we consider the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

If the function $f(x)$ is reasonable (for example, continuous) on the inteval $[a,b]$, then the Riemann sums converge to a unique number as $n \to \infty$. That is,

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x \right) = \int_a^b f(x)\,dx.$$

To define the integral of $f(x, y)$ over a rectangle R, we also define Riemann sums.

Theorem. Suppose the function $f(x, y)$ is continuous on the rectangle R. Then its Riemann sums converge to a unique number as both m and n tend to infinity. That is,

$$\lim_{m,n\to\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta x\, \Delta y \right) = \iint_R f(x, y)\, dA.$$

Example. The animation that I made to illustrate this limiting process involves the function

$$f(x,y) = 2 - \tfrac{1}{2}(x-2)^2 - \tfrac{1}{2}(y-2)^2$$

and the rectangle

$$R = \left\{(x,y) \mid \tfrac{1}{2} \le x \le 3,\ 1 \le y \le 3\right\}.$$

By the end of next class, you will be able to calculate this integral exactly, and you will get

$$\iint_R f(x,y)\, dA = \frac{185}{24} \approx 7.70833.$$

Often in applications, one has data rather than a formula for the function. In that case, Riemann sums can be used to approximate the value of the integral.

The Midpoint Rule

The Midpoint Rule uses the midpoints of the subrectangles as test points.

Example.
A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table below. Use the Midpoint Rule to estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4