Quick review of the Riemann sum definition of the double integral

To define the integral of $f(x, y)$ over a rectangle R, we first define Riemann sums. Let R be the rectangle in the xy-plane defined by

$$R = \{(x, y) \,|\, a \leq x \leq b,\ c \leq y \leq d\}.$$

We subdivide it using m subdivisions along the x-axis and n subdivisions along the y-axis.

Then we pick a test point (x_i^*, y_j^*) from each subrectangle. The Riemann sum associated to this choice of test points is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta x\, \Delta y.$$

Theorem. Suppose the function $f(x, y)$ is continuous on the rectangle R. Then its Riemann sums converge to a unique number as both m and n tend to infinity. That is,

$$\lim_{m,n\to\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta x\, \Delta y \right) = \iint_R f(x, y)\, dA.$$

Example. The animation that I made to illustrate this limiting process involves the function

$$f(x,y) = 2 - \tfrac{1}{2}(x-2)^2 - \tfrac{1}{2}(y-2)^2$$

and the rectangle

$$R = \left\{(x,y) \mid \tfrac{1}{2} \le x \le 3,\ 1 \le y \le 3\right\}.$$

By the end of next class, you will be able to calculate this integral exactly, and you will get

$$\iint\limits_R f(x,y)\, dA = \frac{185}{24} \approx 7.70833.$$

Often in applications, one has data rather than a formula for the function. In that case, Riemann sums can be used to approximate the value of the integral.

The Midpoint Rule

The Midpoint Rule uses the midpoints of the subrectangles as test points.

Example.
A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table below. Use the Midpoint Rule to estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

Properties of the integral

Theorem. The integral satisfies some basic but important properties.

1. $\iint\limits_R f(x,y) + g(x,y)\, dA = \iint\limits_R f(x,y)\, dA + \iint\limits_R g(x,y)\, dA$
2. $\iint\limits_R c\, f(x,y)\, dA = c \iint\limits_R f(x,y)\, dA$ if c is a constant
3. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
$$\iint\limits_R f(x,y)\, dA \geq \iint\limits_R g(x,y)\, dA.$$

Evaluating double integrals via iterated integration

Example. Consider
$$\iint\limits_R \frac{y}{3}\, dA$$
where $R = \{(x,y) \,|\, 0 \leq x \leq 2,\ 0 \leq y \leq 3\}$.

We can generalize this technique to obtain a method for calculating double integrals. Consider a positive function $f(x, y)$ and a rectangle

$$R = \{(x, y) \,|\, a \leq x \leq b,\ c \leq y \leq d\}$$

in the xy-plane. We can calculate the volume of the solid determined by R and $f(x, y)$ using x-slices or y-slices.

Pictures of x-slices and y-slices are available on the course web site and in your textbook as Figures 1 and 2 on p. 839.

The basic idea is to compute the volume of the solid in question by integrating the areas of the slices. For example, suppose that we slice up the solid using y-slices. Let $A(y)$ denote the area of a y-slice. Then

$$\iint\limits_R f(x, y)\, dA = \int_c^d A(y)\, dy.$$

Moreover, for any given y, the area of the y-slice is

$$A(y) = \int_a^b f(x, y)\, dx.$$

We obtain an *iterated integral* that yields the volume of the solid. That is,

$$\iint\limits_R f(x, y)\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy.$$

We compute the inside integral treating y as a constant ("partial integration"), and then we compute the outside integral which only depends on y.

Of course, it also possible to use x-slices rather than y-slices.

Theorem. (Fubini's Theorem) If $f(x, y)$ is continuous on the rectangle

$$R = \{(x, y) \,|\, a \leq x \leq b,\ c \leq y \leq d\},$$

then

$$\begin{aligned}\iint\limits_R f(x, y)\, dA &= \int_a^b \int_c^d f(x, y)\, dy\, dx \\ &= \int_c^d \int_a^b f(x, y)\, dx\, dy.\end{aligned}$$

Let's return to the example discussed earlier.

Example. Consider

$$\iint_R \frac{y}{3}\, dA$$

where $R = \{(x, y) \,|\, 0 \leq x \leq 2,\ 0 \leq y \leq 3\}$.

The calculation that we did involved x-slices.

We can also calculate the integral using y-slices.

Example. Consider the double integral

$$\iint_R x\cos(xy)\,dA$$

where $R = \{(x,y)\,|\,0 \le x \le \pi/4,\ 0 \le y \le 2\}$.

Example. Calculate the average value of the function

$$f(x,y) = xe^{xy}$$

over the rectangle $R = \{(x,y) \,|\, 0 \leq x \leq 1,\ 0 \leq y \leq 2\}$.