Limits and continuity

In order to be able to do calculus for multivariable functions, we need to be able to talk about limits.

Informal definition. We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if $f(x,y) \to L$ as $(x,y) \to (a,b)$ along all paths in the xy-plane.

Here are two examples to illustrate some of the issues that arise.

Example. Consider

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

Example. Consider

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}.$$

Partial derivatives

Consider a function of two variables $f(x, y)$. How do we talk about its rate of change at a given point?

Definition. The partial derivative of $f(x, y)$ in the x-direction at the point (a, b) is defined by

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

In other words, we vary x but keep y constant as we take the limit.

Example. Consider $f(x, y) = 9 - x^2 - y^2$. Let's calculate

$$\frac{\partial f}{\partial x}(1, 2)$$

directly from this definition.

There is another, more efficient way to calculate this partial derivative.

Let's try a more complicated example.

Example. Consider $g(x, y) = y \ln(xy) + y$.

The partial derivative with respect to y is defined in a similar fashion.

Definition. The partial derivative of $f(x, y)$ in the y-direction at the point (a, b) is defined by

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

We keep x constant and vary y as we take the limit.

Example. Consider $g(x, y) = y \ln(xy) + y$ again and calculate $\partial g/\partial y$ this time.

Example. Consider the function $f(x, y) = 9 - x^2 - y^2$ at the point $(1, 2)$. In what direction, the x-direction or the y-direction, does $f(x, y)$ decrease most rapidly?