Using polar coordinates to calculate double integrals

Some important double integrals involve a lot of radial symmetry. Consequently, they are easier to evaluate if we use polar coordinates.

Two basic facts.

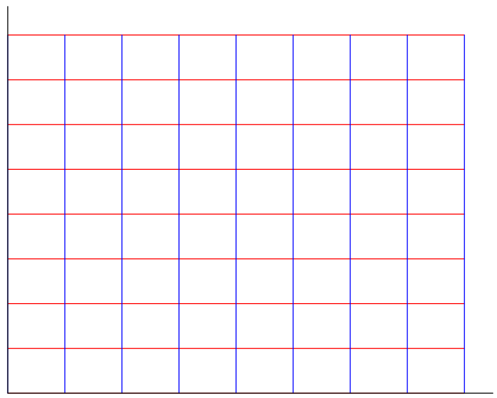
1. $\displaystyle\int_0^1 \sqrt{1-u^2}\,du = \frac{\pi}{4}$

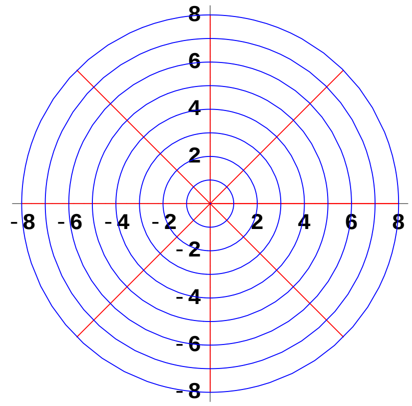
2. The volume of a solid sphere of radius 1 is $\frac{4}{3}\pi$.

Nonexample. Let's try to calculate the volume of a hemisphere of radius 1 using polar coordinates.

What's wrong with this calculation?

The problem with this calculation is that transforming to polar coordinates distorts area.





Recall the formula for area inside a polar curve (see the September 24 handout).

$$\text{area} = \int_{\theta_1}^{\theta_2} \frac{(r(\theta))^2}{2}\, d\theta.$$

Rewriting this formula as a double integral suggests the correct approach for converting double integrals to polar coordinates.

Differential form for area in terms of polar coordinates

$$dA = r\,dr\,d\theta = r\,d\theta\,dr$$

Example. Let's recalculate the volume of the hemisphere using this area adjustment factor.

Example. Calculate the volume of the region that is inside the sphere

$$x^2 + y^2 + z^2 = 4$$

and above the plane $z = 1$.