Two applications of the double integral

Double integrals are used to compute many different types of quantities—not just planar area and volume. Today we will discuss two different applications.

Centroids of lamina

Consider a flat plate (a lamina) with uniform density that occupies a region R in the plane. The center of mass of such an object is called its centroid. We can use double integrals to compute the centroid $(\overline{x}, \overline{y})$. The formulas are

$$\overline{x} = \frac{\displaystyle\iint_R x\, dA}{\displaystyle\iint_R dA} \qquad \text{and} \qquad \overline{y} = \frac{\displaystyle\iint_R y\, dA}{\displaystyle\iint_R dA}.$$

I am not going to go through the derivations of these formulas, but you can read about them in Section 12.5 of your text (pp. 857–859). The discussion in Section 12.5 includes a density function that varies over the region, but I am assuming that this density function is constant. In this case, we get the formulas given above. (You should check me on this.)

Your textbook also has information about moments and centers of mass on pp. 476–479. It is a good exercise for you to derive the formulas given in box 12 on p. 479 using double integrals.

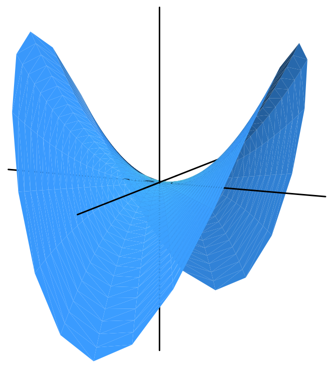
Example. Calculate the centroid of the region in the xy-plane that is bounded by the lines $x = -1$, $y = -1$, $x = 1$, and the graph of $y = x^2$.

Note that the area of this region is $\displaystyle\int_{-1}^{1}\int_{-1}^{x^2} 1\, dy\, dx = \frac{8}{3}$.

Surface area

Another important application of double integrals is the calculation of surface area. For example, suppose we want to calculate the area of the saddle, that is, the area of the surface $z = y^2 - x^2$ over some region in the xy-plane such as the unit disk

$$R = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$



Rather than derive a formula for surface area, we start with a definition.

Definition. Let S be the surface $z = f(x, y)$ where the points (x, y) come from a given region R in the xy-plane. Then

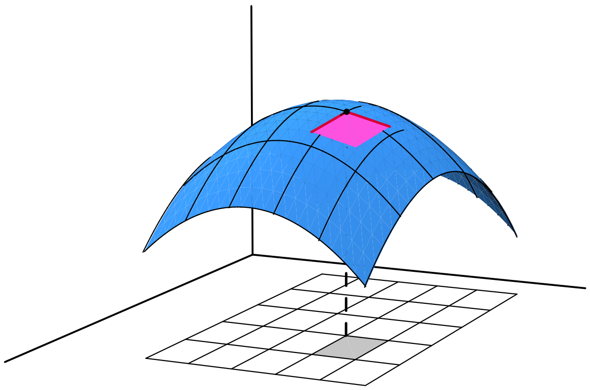
$$\text{Area}(S) = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dA.$$

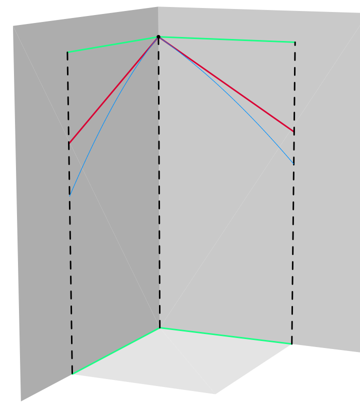
We make this formula a definition because there are many ways to derive wrong formulas, and they usually involve very subtle errors. An advanced course in analysis would include an explanation of why this formula is correct and why the others are wrong, but I would like to give a brief justification to motivate the formula.

Consider a surface S that is the graph of $z = f(x, y)$ over a rectangular region R in the xy-plane. Subdivide R into subrectangles R_{ij} whose areas are the product $(\Delta x)(\Delta y)$, and let S_{ij} be the part of the surface that lies over the subrectangle R_{ij}. Then

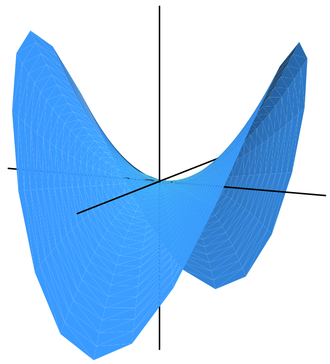
$$\text{area}(S) = \sum_{i,j} \text{area}(S_{ij}).$$

We approximate the areas of the S_{ij} by areas of parallelograms P_{ij} in space (see figure on next page).





Example. Let's find the area of the portion of the saddle $z = y^2 - x^2$ that projects onto the unit circle $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$.



The integral formula for surface area can be expressed in differential notation. If we use the variable S to represent surface area, then the differential formulation of our integral formula is

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

if the surface is the graph of $z = f(x, y)$. We should not forget that this is equivalent to the formula

$$dS = |\mathbf{N}|\, dA$$

where the vector

$$\mathbf{N} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface.

Your textbook also has a more general definition of surface area for parametric surfaces (Definition 4 on p. 868). You should convince yourself that the definition that we discussed today is consistent with that formula.