The differential of surface area

Last class we discussed the surface area formula in the case where the surface S is the graph of a function $z = f(x, y)$ over a region R in the xy-plane.

Definition. Let S be the surface $z = f(x, y)$ where the points (x, y) come from a given region R in the xy-plane. Then

$$\text{Area}(S) = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dA.$$

Whenever we have an integral formula for computing a quantity such as surface area or arc length, we can express that formula in terms of "differentials" (see the September 24 handout for more details regarding differentials).

For surface area, we also use the variable S to represent surface area, and the differential formulation of our integral formula is

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\; dA$$

if the surface is the graph of $z = f(x, y)$. We should not forget that this is equivalent to the formula

$$dS = |\mathbf{N}|\, dA$$

where the vector

$$\mathbf{N} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface.

Your textbook also has a more general definition of surface area for parametric surfaces (Definition 4 on p. 868). You should convince yourself that the definition above is consistent with that formula.

Triple integrals

Integrating functions of three variables is very similar to integrating functions of two variables. Intuitively,

$$\iiint_R f(x,y,z)\,dV = \begin{cases} 1. & \text{“sum” of } f(x,y,z) \text{ over } R \\ 2. & (\text{average value of } f(x,y,z) \text{ over } R)(\text{volume}(R)) \\ 3. & \text{four-dimensional “volume” under the graph of } f(x,y,z) \end{cases}$$

More precisely, the triple integral is a limit of Riemann sums. We partition the region R in space into small rectangular parallelopipids.

For each little "cube" R_{ijk}, we choose a point P_{ijk} in it and then we calculate the sum

$$S_{lmn} = \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(P_{ijk})(\Delta x)(\Delta y)(\Delta z).$$

As l, m, and $n \to \infty$, S_{lmn} approaches a limiting value

$$\iiint_R f(x,y,z)\,dV,$$

which we call the triple integral of the function $f(x,y,z)$ over the region R. This limit is independent of the manner in which we choose the "test" points P_{ijk}.

We calculate this limit by repeated integration just as we did to calculate double integrals.

Intuitively, triple integrals are no more complicated than double integrals, yet they are harder to compute because they are harder to set up. The difficulty comes when we try to set up these integrals over regions more general than rectangular boxes.

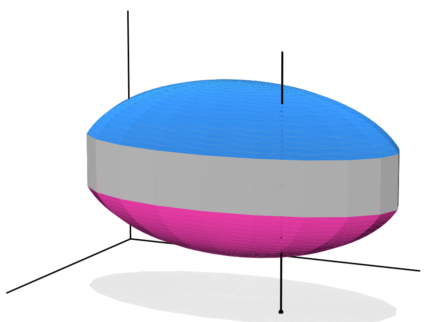
Example. Evaluate

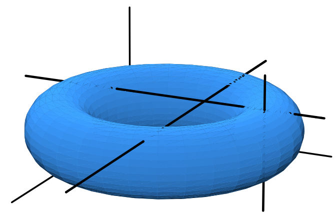
$$\iiint_Q z\, dV$$

where Q is the region bounded by the cylinder $x^2 + z^2 = 9$, the plane $y + z = 3$, and the plane $y = 0$.

Definition. A solid region in space is z-simple if every vertical line that intersects the region enters and exits the region exactly once.

There are analogous definitions of x-simple and y-simple regions.





There is a version of Fubini's Theorem for triple integrals.

Theorem. (Fubini's Theorem) If Q is a z-simple region, then

$$\iiint_Q f(x,y,z)\,dV = \iint_{Q'} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right)\,dA,$$

where Q' is the projection of Q onto the xy-plane and A is area in the xy-plane ($dA = dx\,dy$ or $dA = dy\,dx$).

Now back to the example at hand.

Example. Evaluate

$$\iiint_Q z\,dV$$

where Q is the region bounded by the cylinder $x^2 + z^2 = 9$, the plane $y + z = 3$, and the plane $y = 0$.

We can also view this region as x-simple and evaluate the integral that way. Try setting up the integral yourself. (Answer at the bottom of the next page.)

$$\int \int \quad \int \qquad z\,dx\,dz\,dy$$

Limits of integration for the triple integral at the bottom of p. 5:

$$\int_0^6 \int_{-3}^{3-y} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} z \, dx \, dz \, dy$$