Vector analysis and vector fields

Now we are ready to pursue the final topic of the course—the calculus of vector fields. Before I start introducing the new integrals that we will study, I want to be sure that you understand the notation involved and the typical applications. Therefore, let's start with a few examples that I will use throughout the remainder of the course.

Example. Gravitational Vector Field of Earth: Any mass in space is attracted to the Earth by a gravitational force and Newton's law specifies the strength of the attraction.

Newton's Law:

> Gravitational attraction of a particle of mass m that is a distance d from the center of the Earth is
> $$\frac{GMm}{d^2}$$
> where G is the gravitational constant and M is the mass of the Earth.

Example. Steady fluid flow: For example, consider water slowly flowing through a pipe. The motion of a particle can be represented by a streamline. In other words, it is realistic to assume that the velocity of a particle at a certain point in the pipe is independent of time. Then the fluid flow is entirely determined by the velocity field $\mathbf{V}$.

Example. Electric Field of a Dipole: Two opposite, but equal, point charges placed at $(1, 0)$ and $(-1, 0)$ induce an electric field on the xy-plane. We can calculate this field using Coulomb's law.

Coulomb's Law:

> The force caused by a point charge on a unit charge at a distance d is
>
> $$\pm\frac{k}{d^2}\mathbf{U}$$
>
> where $\mathbf{U}$ is a unit vector pointing in the direction of the point charge, k is a constant, and the sign depends on the nature of the charge. It is positive if the two charges are opposite.

The total electric field caused by the dipole is the vector sum of the two individual electric fields caused by the two different charges. In other words,

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^-,$$

where $\mathbf{E}^+$ represents the force due to the point with positive charge and $\mathbf{E}^-$ represents the force due to the point with negative charge.

Definition. A *vector field* $\mathbf{F}$ in two or three dimensions is a function that associates a vector to each point. In two dimensions, we have

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j},$$

and in three dimensions, we have

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

Example. The vector field

$$\mathbf{V}(x, y) = \omega(-y\mathbf{i} + x\mathbf{j})$$

is the velocity field of any circular motion around the origin with constant angular velocity ω.

Vector fields that are gradients of functions are particularly nice both mathematically and physically.

Definition. A gradient vector field is one that is the gradient of a function. That is,

$$\mathbf{F} = \nabla f.$$

For a gradient vector field $\mathbf{F}(x, y)$ in the xy-plane, we have

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j}.$$

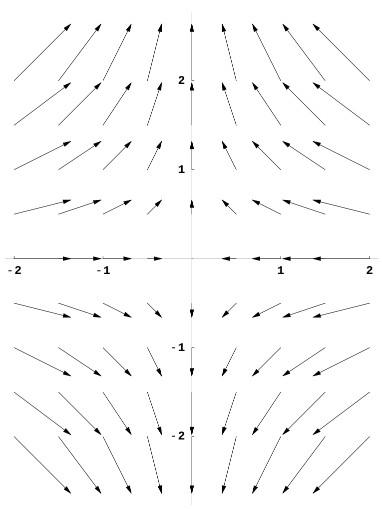
For a gradient vector field $\mathbf{F}(x, y, z)$ in xyz-space, we have

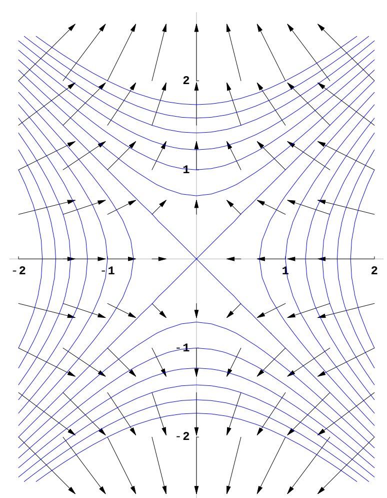
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}\right)\mathbf{k}.$$

Keep the following figures in mind when dealing with gradient vector fields.

Example. (October 22 handout) Consider the function $f(x, y) = \frac{1}{4}(y^2 - x^2)$ and its gradient vector field

$$\mathbf{F}(x, y) = \nabla f(x, y) = \left(-\frac{x}{2}\right)\mathbf{i} + \left(\frac{y}{2}\right)\mathbf{j}.$$





The figure on the left is the gradient vector field alone while the figure on the right has the field superimposed on the level sets of $f(x, y)$.

Example. Here are 8 equations of vector fields and 4 scaled vector fields. Pair each vector field with its corresponding equation.

1. $\mathbf{F}(x,y) = -x\mathbf{i} - 2y\mathbf{j}$
2. $\mathbf{F}(x,y) = -2x\mathbf{i} - y\mathbf{j}$
3. $\mathbf{F}(x,y) = (x^2-1)\mathbf{i} - y\mathbf{j}$
4. $\mathbf{F}(x,y) = -x\mathbf{i} + (y^2-1)\mathbf{j}$
5. $\mathbf{F}(x,y) = (1-y)\mathbf{i} + (1+x)\mathbf{j}$
6. $\mathbf{F}(x,y) = (-1+y)\mathbf{i} - (1+x)\mathbf{j}$
7. $\mathbf{F}(x,y) = -(x+2y)\mathbf{i} + y\mathbf{j}$
8. $\mathbf{F}(x,y) = x\mathbf{i} + (2x-y)\mathbf{j}$

The four vector fields: