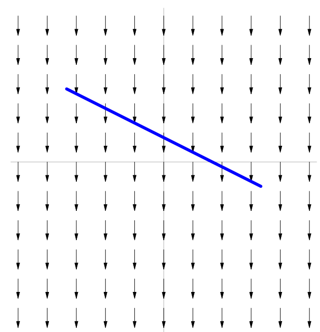
Line integrals

Last class we discussed path integrals. These are integrals with which we add up a scalar quantity over a curve in the plane or in space. Today we discuss an important special case of a path integral—the line integral.

Suppose that we have a vector field and a curve and we want to measure how much of the field is pointing in the direction of the curve. If the field is a gravitational field, then this quantity is the work done by the gravitational force in displacing a particle along the curve. If the field is a velocity field of a fluid, then this quantity is the circulation of the fluid along the curve. If the field is an electric field and the curve is closed, then this quantity is related to current passing through any surface bounded by the curve (Ampère's Law).

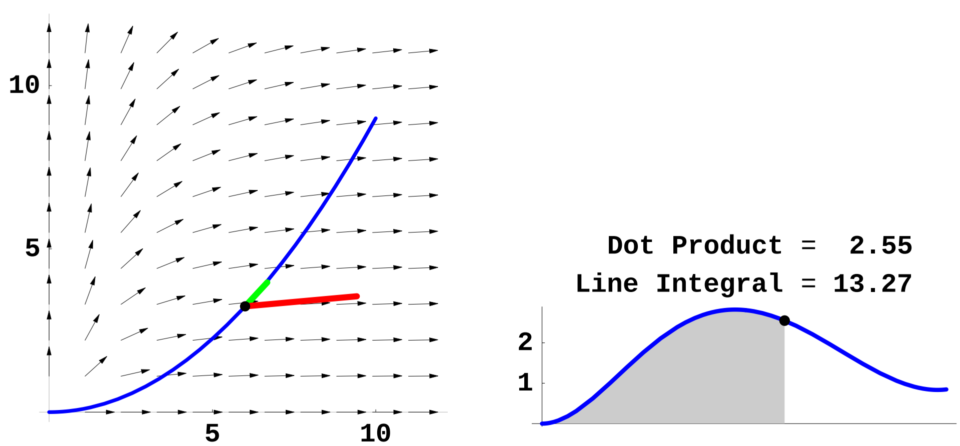
Example. Consider a constant force field $\mathbf{F}(x, y) = \mathbf{C}$ and a linear displacement $\mathbf{D}$.



Definition. Consider a vector field $\mathbf{F}(x, y)$ and a curve C. The line integral of $\mathbf{F}(x, y)$ along C is the path integral

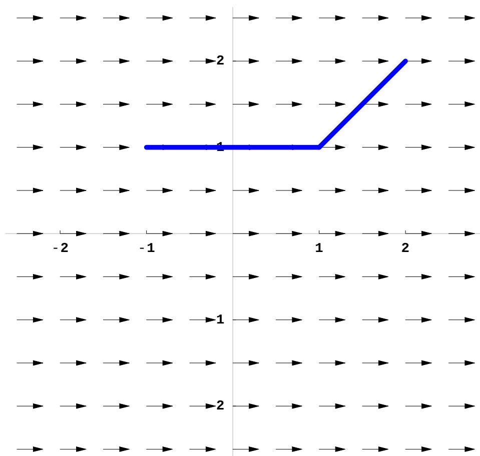
$$\int_C (\mathbf{F} \cdot \mathbf{T})\, ds,$$

where $\mathbf{T}$ is the unit tangent vector along the curve C.



Let's eyeball three examples before we start the computations.

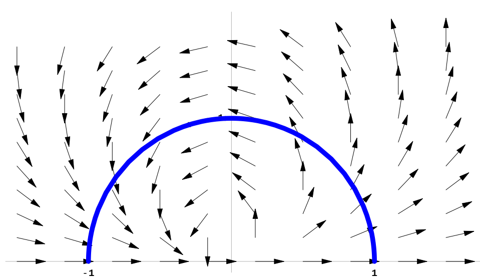
Example 1. Let $\mathbf{F}(x,y) = \mathbf{i}$ and C consists of two line segments—one from $(-1,1)$ to $(1,1)$ and another from $(1,1)$ to $(2,2)$.



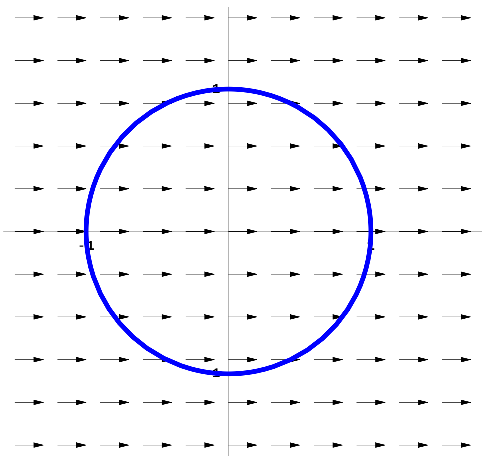
Example 2. Let

$$\mathbf{F}(x,y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$$

and C consists of the top half of the positively-oriented unit circle centered at the origin.



Example 3. Let $\mathbf{F}(x, y) = \mathbf{i}$ and C be the positively-oriented unit circle centered at the origin.

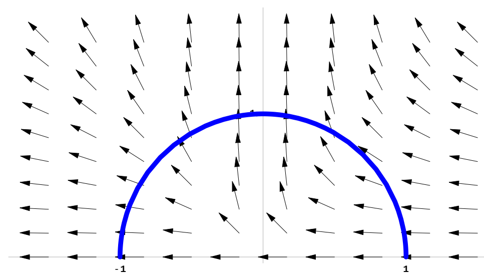


How do we go about calculating a line integral?

Suppose that the curve C is parameterized by a vector-valued function $\mathbf{r}(t)$ from $t = a$ to $t = b$. Then

$$\begin{aligned}\int_C (\mathbf{F} \cdot \mathbf{T})\, ds &= \int_a^b \left(\mathbf{F} \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)|\, dt \\ &= \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt \\ &= \int_C \mathbf{F} \cdot\, d\mathbf{r}.\end{aligned}$$

Example. Let $\mathbf{F}(x, y) = -x^2\mathbf{i} + y^2\mathbf{j}$ and C be the top half of the positively-oriented unit circle centered at the origin. Calculate the line integral of $\mathbf{F}$ along C.



Example. What integral do we have to calculate to determine the value of the line integral in Example 2 above?

Comments on notation: Suppose that the curve C is parameterized by a vector-valued function $\mathbf{r}(t)$ from $t = a$ to $t = b$. Then

$$\int_C (\mathbf{F} \cdot \mathbf{T})\, ds = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt \equiv \int_C \mathbf{F} \cdot d\mathbf{r} \equiv \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}.$$

Some classical notation: In the planar case, the vector field is often written as

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$

Suppose that the curve C is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then

$$\mathbf{r}'(t) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j},$$

and

$$\int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt = \int_a^b \left(P(x, y)\frac{dx}{dt} + Q(x, y)\frac{dy}{dt} \right)\, dt.$$

For this reason, line integrals are often written as

$$\int_C P(x, y)\, dx + Q(x, y)\, dy.$$

In space, we have

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

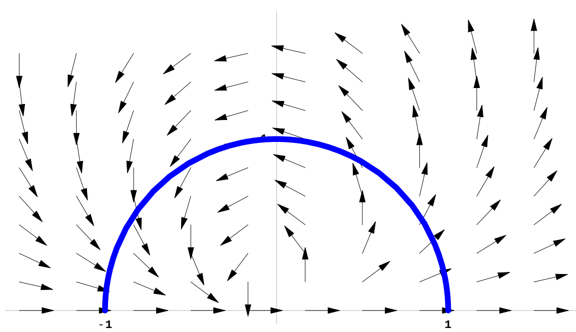
Then the line integral is written as

$$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz.$$

Example. Let's revisit Example 2 using the classical notation. Recall that

$$\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$$

and C consists of the top half of the positively-oriented unit circle centered at the origin.



In the classical notation, we write this line integral as

$$\int_C (x^2 - y^2)\, dx + 2xy\, dy.$$