More on Green’s Theorem

Green’s Theorem relates line integrals of vector fields in the $xy$-plane to double integrals.

**Theorem.** (Green’s Theorem) Let $C$ be a positively-oriented, simple, closed curve in the plane and let $D$ denote the region it encloses. Then

$$\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$ 

**Example.** Let $C$ be the perimeter of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Calculate

$$\oint_C x \, dx + xy \, dy.$$ 

**Note:** If $\mathbf{F}(x,y) = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j}$ has a potential function, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0,$$

and we see that $\oint_C P \, dx + Q \, dy = 0$. 

1
Example. Compute the line integral

\[ \oint -y^3 \, dx + x^3 \, dy \]

over the unit circle in the positively-oriented direction.
The vector forms of Green’s Theorem

I would like to use Green’s Theorem to explain two basic concepts in vector analysis—the curl and divergence of a vector field—in the case where the vector field $\mathbf{F}$ is a planar vector field. It helps if you consider the vector field $\mathbf{F}$ as a velocity field of a fluid. In this case, imagine the flowlines or streamlines of the fluid.

For velocity fields of fluids, the line integral $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ is called the *circulation* of the fluid along the curve.

Here are pictures of three examples:
**Definition.** For a planar vector field \( \mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \), the curl of \( \mathbf{F}(x, y) \) is the vector field

\[
\text{curl} \mathbf{F}(x, y) = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.
\]

To interpret the curl of \( \mathbf{F} \) in this situation, we use Green’s Theorem.

**Theorem.** (The vector form of Green’s Theorem) Let \( C \) be a positively-oriented, simple, closed curve in the \( xy \)-plane and let \( D \) be the region that is enclosed by \( C \). Then

\[
\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_D (\text{curl} \mathbf{F}) \cdot \mathbf{k} \, dA.
\]

How does this help us interpret the curl of \( \mathbf{F} \)?

---

**Divergence for planar vector fields**

**Definition.** Given a vector field in the plane \( \mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \), the divergence of \( \mathbf{F} \) is the scalar field (scalar function) defined by

\[
\text{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.
\]

To understand what divergence measures in the case of our velocity field of a fluid, we consider a different path integral. Given a simple, closed curve \( C \) in the plane, consider the path integral

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds,
\]

where \( \mathbf{n} \) is a unit normal vector to \( C \) that points outside the region enclosed by \( C \).
To see that this path integral is also a line integral, we need to recall two facts from earlier in the semester:

1. Suppose that the curve $C$ is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then we can reparametrize $C$ using arc length $s$ as the parameter (see pp. 709–710 in our text). Then the unit tangent vector

$$
\mathbf{T} = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j}.
$$

2. Because the curve $C$ is positively oriented, we get the unit normal vector $\mathbf{n}$ by rotating $\mathbf{T}$ by $\pi/2$ radians in the clockwise direction. Given any vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in the plane, then the vector perpendicular to $\mathbf{u}$ rotated by $\pi/2$ radians in the clockwise direction is $u_2\mathbf{i} - u_1\mathbf{j}$.

Theorem. (planar Divergence Theorem) Let $C$ be a positively-oriented, simple, closed curve in the $xy$-plane and let $D$ be the region that is enclosed by $C$. Then

$$
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA = \iint_D \left( \text{div} \, \mathbf{F} \right) \, dA.
$$

This identity justifies the name “divergence.”