More on Green's Theorem

Green's Theorem relates line integrals of vector fields in the xy-plane to double integrals.

Theorem. (Green's Theorem) Let C be a positively-oriented, simple, closed curve in the plane and let D denote the region it encloses. Then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Example. Let C be the perimeter of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Calculate

$$\oint_C x\,dx + xy\,dy.$$

Note: If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ has a potential function, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0,$$

and we see that $\oint_C P\,dx + Q\,dy = 0$.

Example. Compute the line integral

$$\oint -y^3\,dx + x^3\,dy$$

over the unit circle in the positively-oriented direction.

The vector forms of Green's Theorem

I would like to use Green's Theorem to explain two basic concepts in vector analysis—the curl and divergence of a vector field—in the case where the vector field $\mathbf{F}$ is a planar vector field. It helps if you consider the vector field $\mathbf{F}$ as a velocity field of a fluid. In this case, imagine the flowlines or streamlines of the fluid.

For velocity fields of fluids, the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is called the *circulation* of the fluid along the curve.

Here are pictures of three examples:

Definition. For a planar vector field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, the curl of $\mathbf{F}(x,y)$ is the vector field

$$\operatorname{curl}\mathbf{F}(x,y) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

To interpret the curl of $\mathbf{F}$ in this situation, we use Green's Theorem.

Theorem. (The vector form of Green's Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA.$$

How does this help us interpret the curl of $\mathbf{F}$?

Divergence for planar vector fields

Definition. Given a vector field in the plane $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

To understand what divergence measures in the case of our velocity field of a fluid, we consider a different path integral. Given a simple, closed curve C in the plane, consider the path integral

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds,$$

where $\mathbf{n}$ is a unit normal vector to C that points outside the region enclosed by C.

To see that this path integral is also a line integral, we need to recall two facts from earlier in the semester:

1. Suppose that the curve C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then we can reparametrize C using arc length s as the parameter (see pp. 709–710 in our text). Then the unit tangent vector
$$\mathbf{T} = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j}.$$

2. Because the curve C is positively oriented, we get the unit normal vector $\mathbf{n}$ by rotating $\mathbf{T}$ by $\pi/2$ radians in the clockwise direction. Given any vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in the plane, then the vector perpendicular to $\mathbf{u}$ rotated by $\pi/2$ radians in the clockwise direction is $u_2\mathbf{i} - u_1\mathbf{j}$.

Theorem. (planar Divergence Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint\limits_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint\limits_D (\operatorname{div}\mathbf{F})\ dA.$$

This identity justifies the name "divergence."