Correction to the computation at end of class on Monday

At the end of class on Monday, I used Green's Theorem to calculate the path integral

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is a unit normal vector to C that points outside the region enclosed by C.

If the curve C is parametrized by arc length s, then

$$\mathbf{n} = \left(\frac{dy}{ds}\right) \mathbf{i} - \left(\frac{dx}{ds}\right) \mathbf{j},$$

and if $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$, we get

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds &= \oint_C (P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}) \cdot \left(\left(\frac{dy}{ds}\right) \mathbf{i} - \left(\frac{dx}{ds}\right) \mathbf{j} \right) ds \\
&= \oint_C P(x, y)\, dy - Q(x, y)\, dx \\
&= \iint\limits_D \left(\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y} \right) \right) dA.
\end{aligned}$$

This result is called the planar Divergence Theorem.

Theorem. (planar Divergence Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint\limits_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint\limits_D (\operatorname{div} \mathbf{F}) \; dA.$$

This identity justifies the name "divergence."

Curl and divergence of vector fields in space

Last class we discussed curl and divergence for planar vector fields. The curl of a vector field measures how much it "spirals." The divergence of a vector field measures how fast it "expands." Now let's see how they are defined for vector fields in space.

Definition. Given a vector field in space

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k},$$

the curl of $\mathbf{F}$ is another vector field defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

This formula seems quite nasty, but there is an easy way to remember it.

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}.$$

Example. Calculate $\operatorname{curl}\mathbf{F}$ for $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} + yz^2\,\mathbf{j} + x^2z\,\mathbf{k}$.

The formula for the curl is related to the question of whether a vector field in space has a potential function.

Theorem. If the vector field $\mathbf{F}(x, y, z)$ in space is the gradient of some function $f(x, y, z)$, then

$$\operatorname{curl} \mathbf{F} = \mathbf{0}.$$

Sometimes you will see this fact expressed very succinctly as

$$\nabla \times \nabla f = \mathbf{0}.$$

The theorem gives a necessary condition for a vector field in space to have a potential function. As in the case of vector fields in the plane, this necessary condition is also sufficient if the vector field is continuously differentiable everywhere.

Theorem. Suppose the vector field $\mathbf{F}$ is defined and continuously differentiable for all (x, y, z). If

$$\operatorname{curl} \mathbf{F} = \mathbf{0}$$

for all (x, y, z), then $\mathbf{F}$ has a potential function.

The potential function is calculated using a generalization of the procedure that we discussed on November 30.

There is an animation of the concept of curl from the University of Minnesota that is referenced on the class web site.

Divergence

Definition. Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Shorthand notation: $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$.

Example. Calculate $\operatorname{div} \mathbf{F}$ for $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + yz^2\,\mathbf{j} + x^2 z\,\mathbf{k}$.