Surface integrals

A path integral is how we "add up" a function over a curve in the plane or in space. Similarly, a surface integral is how we add up a function $f(x, y, z)$ over a surface in space.

We want an integral such that

$$\iint_S f(x,y,z)\,dS = \text{ the "sum" of } f(x,y,z) \text{ over the surface } S$$
$$= (\text{the average of } f(x,y,z) \text{ over } S)(\text{surface area}(S)).$$

Special case: Assume that the surface S is the graph of a function $g(x, y)$ over a region R in the xy-plane. In this case, the differential of surface area is

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dA.$$

This differential can also be expressed as

$$dS = |\mathbf{N}|\,dA,$$

where the vector

$$\mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface (see the handouts for November 5 and 7.).

Consequently, we have

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dA.$$

Example. Let S be the portion of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. Let's find its center of mass.

If the center of mass is denoted $(\overline{x}, \overline{y}, \overline{z})$, then we know that $\overline{x} = \overline{y} = 0$. Also,

$$\overline{z} = \frac{\displaystyle\iint_S z\, dS}{\text{area}(S)}.$$

Flux integrals

Given a vector field in space

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

and a surface S, the flux of $\mathbf{F}$ across S is the surface integral

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS,$$

where $\mathbf{n}$ is the unit normal vector at each point of S.

Special case: Again we assume that the surface S is the graph of a function $g(x,y)$ over a region R in the xy-plane. Note that

$$\mathbf{n} = \pm\frac{\mathbf{N}}{|\mathbf{N}|} \text{ where } \mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}.$$

In this case, we compute the flux of $\mathbf{F}$ across S by

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \pm\iint_R \left(\mathbf{F}\cdot\frac{\mathbf{N}}{|\mathbf{N}|}\right)|\mathbf{N}|\, dA$$

$$= \pm\iint_R (\mathbf{F}\cdot\mathbf{N})\, dA.$$

Example. Consider the surface S that is the boundary of the solid that is bounded by the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy-plane. Also, consider the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 3\mathbf{k}.$$

Let's calculate the flux across S in the direction of the outer normal.

The surface S splits nicely into two surfaces. Let's denote the part of S given by the paraboloid as S_1 and the part of S that lies in the xy-plane as S_2.

Flux across S_1:

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Flux across S_2: