An example of a flux integral

Given a vector field in space

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

and a surface S, the flux of $\mathbf{F}$ across S is the surface integral

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS,$$

where $\mathbf{n}$ is the unit normal vector at each point of S.

In the special case where the surface is the graph of a function $g(x,y)$ over a region R in the xy-plane, we compute the flux of $\mathbf{F}$ across S by

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \pm \iint_R (\mathbf{F} \cdot \mathbf{N})\ dA,$$

where the vector

$$\mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface.

Example. Consider the surface S that is the boundary of the solid that is bounded by the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy-plane. Also, consider the vector field

$$\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 3\mathbf{k}.$$

Let's calculate the flux across S in the direction of the outer normal.

The surface S splits nicely into two surfaces. Let's denote the part of S given by the paraboloid as S_1 and the part of S that lies in the xy-plane as S_2.

Flux across S_1:

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Flux across S_2:

There are two three-dimensional versions of Green's Theorem—Stokes's Theorem and the Divergence Theorem. Stokes's Theorem explains the meaning of curl, and the Divergence Theorem explains the meaning of the divergence of a vector field. We will finish the semester with Stokes's Theorem.

Theorem. (Stokes's Theorem) Let S be an oriented surface that is bounded by a simple, closed curve C with positive orientation. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Example. Let S be the portion of the plane $y+z=1$ that lies inside the cylinder $x^2+y^2=1$. Orient S using $\mathbf{N} = \mathbf{j} + \mathbf{k}$ and let C be the positively-oriented curve that is the intersection of the plane and the cylinder. Let's calculate the line integral

$$\int_C x^2\, dx + xy^2\, dy + z^3\, dz$$

using Stokes's Theorem.