More on the cross product

Last class we started our discussion of the cross product of two vectors, and we focused on its geometric properties. Today we will derive a formula for the cross product and do some applications.

Example. Last class we computed $(4\mathbf{i}+\mathbf{j}) \times (7\mathbf{i}+2\mathbf{k})$ as follows:

$$\begin{aligned}(4\mathbf{i}+\mathbf{j}) \times (7\mathbf{i}+2\mathbf{k}) &= (4\mathbf{i}) \times (7\mathbf{i}+2\mathbf{k}) + (\mathbf{j}) \times (7\mathbf{i}+2\mathbf{k}) \\ &= 8(\mathbf{i}\times\mathbf{k}) + 7(\mathbf{j}\times\mathbf{i}) + 2(\mathbf{j}\times\mathbf{k}) \\ &= -8\mathbf{j} - 7\mathbf{k} + 2\mathbf{i} \\ &= 2\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}.\end{aligned}$$

To get a general formula, we can repeat the same type of calculation on

$$(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}) \times (b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}).$$

The result is

$$(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

There is a handy way of remembering this formula that uses the determinant of a 3×3 matrix.

We can calculate the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

in one of two ways.

1. Expansion by minors: The determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \alpha_1 \begin{vmatrix} \beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} - \alpha_2 \begin{vmatrix} \beta_1 & \beta_3 \\ \gamma_1 & \gamma_3 \end{vmatrix} + \alpha_3 \begin{vmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}.$$

This method is the one that your textbook describes.

2. Sum of six terms, each a product of three numbers: We can also compute this determinant by writing it as

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 & \beta_3 & \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_1 & \gamma_2 \end{matrix}$$

   Then we get

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = (\alpha_1\beta_2\gamma_3) + (\alpha_2\beta_3\gamma_1) + (\alpha_3\beta_1\gamma_2)$$
$$-(\alpha_3\beta_2\gamma_1) - (\alpha_1\beta_3\gamma_2) - (\alpha_2\beta_1\gamma_3)$$

Using this formula, we can write

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example. We use the determinant formula to calculate $(4\mathbf{i} + \mathbf{j}) \times (7\mathbf{i} + 2\mathbf{k})$.

Applications to Lines and Planes

We have discussed finding the equation of a plane if we are given a point P on the plane and a normal vector $\mathbf{N}$. Now that we can compute the cross product of two vectors, we are able to find the equation of a plane determined in more familiar ways, e.g., determined by three noncollinear points or by two intersecting lines. We use the cross product to determine $\mathbf{N}$.

Example. Find an equation for the plane that contains the three points $P_1 = (1, 1, 1)$, $P_2 = (2, -2, 2)$, and $P_3 = (0, 2, 1)$.

Example. Find the equation of the plane that contains the two lines

$$x = y = z \qquad \text{and} \qquad \frac{x-1}{2} = \frac{y-1}{3} = z - 1.$$

Why do these lines intersect? Where do they intersect? What are their direction vectors $\mathbf{D}_1$ and $\mathbf{D}_2$?

Example. Let ℓ_1 be line through $(1,0,0)$ and $(1,2,2)$. Let ℓ_2 be the line through $(1,1,1)$ and $(1+\sqrt{2},2,2)$. Do these lines intersect? What is the angle of intersection?