Vector-valued functions and parameterized curves

First, let’s quickly review parameterized curves in the plane (see Section 1.7 of your text).

**Definition.** A *parametric curve* in the $xy$-plane is a pair of scalar functions

$$
  x = f(t) \\
  y = g(t)
$$

We trace out the curve by plotting all points of the form

$$
  \text{trace} = \{(f(t), g(t)) | \text{for all } t \text{ in the domains of } f \text{ and } g\}.
$$

**Example.**

$$
  x(t) = 3t + 1 \\
  y(t) = 2t + 2
$$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-2$</td>
<td>$0$</td>
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<tr>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
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<tr>
<td>$1$</td>
<td>$4$</td>
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<td>$2$</td>
<td>$7$</td>
<td>$6$</td>
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We can solve for $t$ to get a nonparametric representation of the curve.

$$
  x = 3t + 1 \\
  x - 1 = 3t \\
  \frac{x - 1}{3} = t
$$

Therefore, we have

$$
  y = 2 \left( \frac{x - 1}{3} \right) + 2 \\
  = \frac{2}{3}x - \frac{2}{3} + 2 \\
  = \frac{2}{3}x + \frac{4}{3}.
$$
**Remark.** Any parameterized equation of the form

\[ x = at + b \]
\[ y = ct + d \]

is a line.

In this course, you need to know how to parameterize any line in the plane. For practice, start with two points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) and form parametric equations for the line that contains \( P_1 \) and \( P_2 \).

Curves in space can be described in essentially the same manner as curves in the plane. Their parametric representation is given by three scalar-valued functions

\[ x = f(t) \]
\[ y = g(t) \]
\[ z = h(t) \]

**Vector-valued functions**

When we study curves in the plane or in space, it is often useful to employ vector techniques, and we do so by using vector-valued functions.

Given a parameterized curve in space of the form

\[ x = f(t) \]
\[ y = g(t) \]
\[ z = h(t) \]

we can combine these three functions to make one vector-valued function

\[ \mathbf{P}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}. \]

The vector \( \mathbf{P}(t) \) is often thought of as a position vector that varies with the parameter \( t \).
Example 1. Let $L(t) = (4 - t) \mathbf{i} + (5t - 1) \mathbf{j} + (3 + \frac{1}{2}t) \mathbf{k}$.

Example 2. Let $H(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j} + t \mathbf{k}$.

Example 3. Let $K(t) = \left( (2 + \cos \frac{3}{2}t) \cos t \right) \mathbf{i} + \left( (2 + \cos \frac{3}{2}t) \sin t \right) \mathbf{j} + \left( \sin \frac{3}{2}t \right) \mathbf{k}$. 
In one-dimensional calculus, we define the derivative of a scalar-valued function as the limit
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]
It is the limit of the change in \( f(x) \) divided by the change in \( x \). We can do the same for vector-valued functions.

**Definition.** Let \( \mathbf{f}(t) \) be a vector-valued function. Then
\[ \mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t + h) - \mathbf{f}(t)}{h}. \]
What does this limit represent? First, let’s consider the definition in terms of motion in space.

![Diagram of a curve]

We see that the secants limit on a tangent vector. We divide by \( h \) to stop the vectors from shrinking to zero. As we shall see on Wednesday, there is another good reason for dividing by \( h \).

Since we have this interesting vector associated to \( \mathbf{f}(t) \), how do we compute it?

**Theorem.** Let \( \mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \). Then \( \mathbf{f}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} \).

**Example.** Consider a curve that is very similar to the circular helix. Let
\[ \mathbf{r}(t) = (\cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + tk. \]

Then
\[ \mathbf{r}'(t) = (-\sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + k. \]
So the derivative at \( t = \pi/3 \) is
\[ \frac{-\sqrt{3}}{2} \mathbf{i} + \mathbf{j} + \mathbf{k}. \]
This vector is tangent to the elliptical helix at \( t = \pi/3 \).
Example. Find the equation of the tangent line to the curve

\[ r(t) = e^t \mathbf{i} + 2 \sin t \mathbf{j} + (t^2 - 2) \mathbf{k}. \]

at the point \((1, 0, -2)\).