Vector-valued functions and parameterized curves

First, let's quickly review parameterized curves in the plane (see Section 1.7 of your text).

Definition. A *parametric curve* in the xy-plane is a pair of scalar functions

$$x = f(t)$$
$$y = g(t)$$

We trace out the curve by plotting all points of the form

$$\text{trace } = \{(f(t), g(t)) \,|\, \text{for all } t \text{ in the domains of } f \text{ and } g\}.$$

Example.

$$x(t) = 3t + 1$$
$$y(t) = 2t + 2$$

t	x	y
−1	−2	0
0	1	2
1	4	4
2	7	6

We can solve for t to get a nonparametric representation of the curve.

$$x = 3t + 1$$

$$x - 1 = 3t$$

$$\frac{x-1}{3} = t$$

Therefore, we have

$$y = 2\left(\frac{x-1}{3}\right) + 2$$

$$= \frac{2}{3}x - \frac{2}{3} + 2$$

$$= \frac{2}{3}x + \frac{4}{3}.$$

Remark. Any parameterized equation of the form

$$x = at + b$$
$$y = ct + d$$

is a line.

In this course, you need to know how to parameterize any line in the plane. For practice, start with two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ and form parametric equations for the line that contains P_1 and P_2.

Curves in space can be described in essentially the same manner as curves in the plane. Their parametric representation is given by three scalar-valued functions

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t).$$

Vector-valued functions

When we study curves in the plane or in space, it is often useful to employ vector techniques, and we do so by using vector-valued functions.

Given a parameterized curve in space of the form

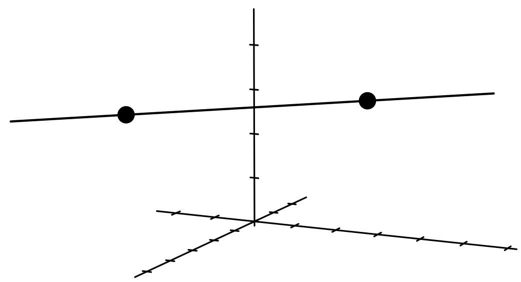
$$x = f(t)$$
$$y = g(t)$$
$$z = h(t),$$

we can combine these three functions to make one vector-valued function

$$\mathbf{P}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

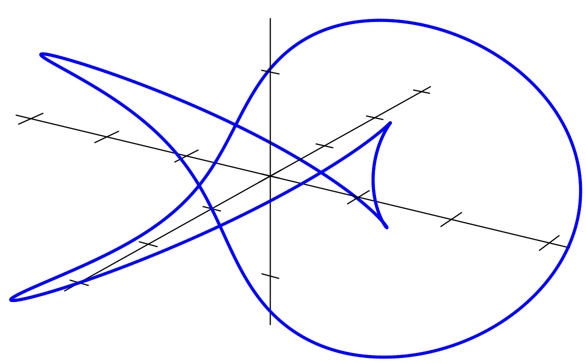
The vector $\mathbf{P}(t)$ is often thought of as a position vector that varies with the parameter t.

Example 1. Let $\mathbf{L}(t) = (4 - t)\,\mathbf{i} + (5t - 1)\,\mathbf{j} + (3 + \frac{1}{2}t)\,\mathbf{k}$.



Example 2. Let $\mathbf{H}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$.

Example 3. Let $\mathbf{K}(t) = \Big((2 + \cos \frac{3}{2}t)\cos t\Big)\ \mathbf{i} + \Big((2 + \cos \frac{3}{2}t)\sin t\Big)\ \mathbf{j} + \Big(\sin \frac{3}{2}t\Big)\ \mathbf{k}$.



In one-dimensional calculus, we define the derivative of a scalar-valued function as the limit

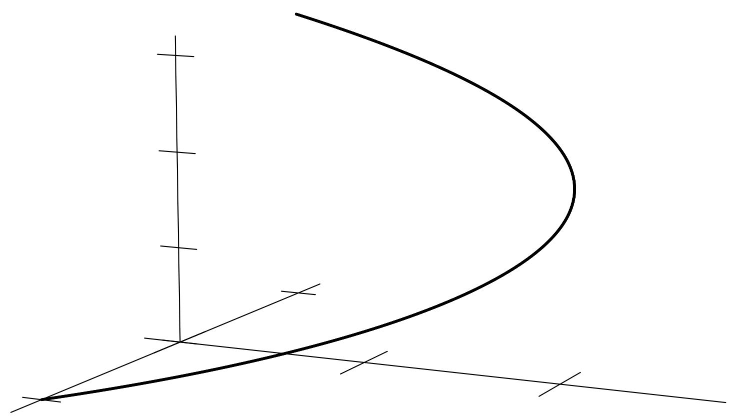
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

It is the limit of the change in $f(x)$ divided by the change in x. We can do the same for vector-valued functions.

Definition. Let $\mathbf{f}(t)$ be a vector-valued function. Then

$$\mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

What does this limit represent? First, let's consider the definition in terms of motion in space.



We see that the secants limit on a tangent vector. We divide by h to stop the vectors from shrinking to zero. As we shall see on Wednesday, there is another good reason for dividing by h.

Since we have this interesting vector associated to $\mathbf{f}(t)$, how do we compute it?

Theorem. Let $\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{f}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Example. Consider a curve that is very similar to the circular helix. Let

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k}.$$

Then

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}.$$

So the derivative at $t = \pi/3$ is

$$-\frac{\sqrt{3}}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

This vector is tangent to the elliptical helix at $t = \pi/3$.

Example. Find the equation of the tangent line to the curve

$$\mathbf{r}(t) = e^t\,\mathbf{i} + 2\sin t\,\mathbf{j} + (t^2 - 2)\,\mathbf{k}.$$

at the point $(1, 0, -2)$.