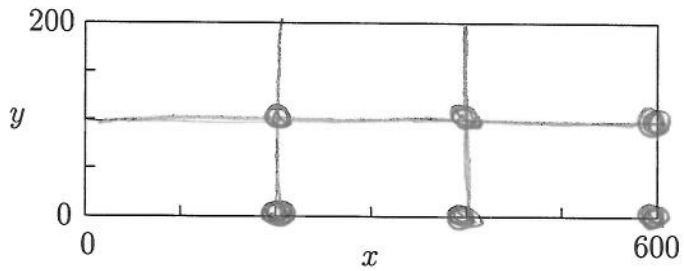
1. (19 points) The table below contains rainfall readings in inches over a 600-foot-by-200-foot rectangular field (600 feet east-west and 200 feet north-south—see a sketch of the field immediately below).



$\Delta x = 200$ ft
$\Delta y = 100$ ft
● = test points

The readings were made over a rectangular grid at regular intervals, and they are recorded in the table using a rectangular coordinate system where the origin represents the southwest corner of the region, x represents the east-west distance of the grid point to the origin, and y represents the north-south distance of the grid point to the origin.

	0	100	200	300	400	500	600
0	3.1	3.2	3.2	3.3	3.5	3.6	3.8
50	3.2	3.3	3.4	3.5	3.6	3.9	3.8
100	3.3	3.5	3.6	3.7	3.8	3.9	3.7
150	3.4	3.6	3.7	3.7	3.8	3.8	3.9
200	3.5	3.5	3.5	3.6	3.7	3.9	4.0

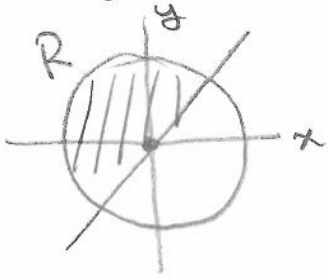
Estimate the total amount of rain that fell on the entire field using a Riemann sum with three east-west subdivisions ($m = 3$) and two north-south subdivisions ($n = 2$). As sample/test points, use the points that are the farthest south and east in each subrectangle. Express your answer in cubic feet.

Riemann Sum $= (3.2 + 3.5 + 3.8 + 3.6 + 3.8 + 3.7)(\Delta x)(\Delta y)$

$= (21.6 \text{ inches})(200 \text{ ft})(100 \text{ ft})$

$= 36{,}000 \text{ ft}^3$

2. (19 points) Calculate the average value of the function $f(x, y) = e^{3x^2+3y^2}$ over the region that is inside the circle $x^2 + y^2 = 5$ and above the line $y = x$.



convert to polar coordinates

$$\iint_R e^{3x^2+3y^2}\,dA = \int_{\pi/4}^{5\pi/4}\int_0^{\sqrt{5}} e^{3r^2}\, r\,dr\,d\theta$$

$u = 3r^2$

$du = 6r\,dr$

$r = 0 \Rightarrow u = 0$

$r = \sqrt{5} \Rightarrow u = 15$

$$= \int_{\pi/4}^{5\pi/4}\int_0^{15} e^u \left(\tfrac{1}{6}\right) du\, d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \left(\frac{e^{15}-1}{6}\right) d\theta = \frac{\pi}{6}\left(e^{15}-1\right)$$

area of region $= \frac{1}{2}(\pi)(5) = \frac{5}{2}\pi$

$$\text{average value} = \frac{\frac{\pi}{6}\left(e^{15}-1\right)}{\frac{5}{2}\pi} = \left(\frac{2}{5}\right)\left(\frac{1}{6}\right)\left(e^{15}-1\right)$$

$$= \frac{e^{15}-1}{15}$$

3. (19 points) The following table gives the heat index I (perceived temperature) as a function of actual temperature T ($90 \le T \le 100$) and relative humidity H ($50 \le H \le 90$).

H

T

	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

(a) Use the table to find a linear approximation of the heat index I when the temperature is near 94° and the relative humidity is near 60%.

estimate $\frac{\partial I}{\partial T} \approx \frac{116-105}{4} = \frac{11}{4} = 2.75$

estimate $\frac{\partial I}{\partial H} \approx \frac{114-107}{10} = \frac{7}{10} = 0.7$

heat index $I \approx 111 + 2.75(T-94) + 0.7(H-60)$

(b) Estimate the heat index when the temperature is 95° and the relative humidity is 58%.

$T = 95^\circ \Rightarrow \Delta T = 1^\circ$

$H = 58\% \Rightarrow \Delta H = -2\%$

heat index $\approx 111 + (2.75)(1) - (0.7)(2)$

$T = 95^\circ$

$H = 58\%$ $= 112.35^\circ$

4. (20 points) Find the points on the hyperboloid of one sheet

$$4x^2 + y^2 - z^2 = 4$$

where the tangent plane is parallel to the plane $2x + 2y + z = 7$.

Let $f(x,y,z) = 4x^2 + y^2 - z^2$. Surface is the level set of level 4.

$\nabla f = (8x)\vec{i} + (2y)\vec{j} + (-2z)\vec{k}$

plane has normal vector $\vec{N} = 2\vec{i} + 2\vec{j} + \vec{k}$

Want points where $\nabla f = \lambda \vec{N}$ for some scalar λ.

$$\Rightarrow 8x = 2\lambda \Rightarrow 4x = y = -2z$$
$$2y = 2\lambda$$
$$-2z = \lambda$$

To find the points,

$$4x^2 + (4x)^2 - (-2x)^2 = 4$$
$$16x^2 = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

Two points:

$(\frac{1}{2}, 2, -1)$ and $(-\frac{1}{2}, -2, 1)$

5. (20 points) Find all critical points of the function $f(x, y) = x^3 + y^3 + 6xy$ and classify them using the Second Partials Test.

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 + 6y = 0 \Rightarrow 3x^2 = -6y \\ \frac{\partial f}{\partial y} = 3y^2 + 6x = 0 \Rightarrow 3y^2 = -6x \end{cases}$$

$x^2 = -2y$ and $y^2 = -2x$

$\Rightarrow x^4 = 4y^2 = 4(-2x)$

$\Rightarrow x^4 + 8x = 0 \Rightarrow x(x^3+8) = 0$

$\Rightarrow x = 0$ or $x = -2$

$\Rightarrow (x,y) = (0,0)$ or $(x,y) = (-2,-2)$

Hessian matrix: $\begin{pmatrix} 6x & 6 \\ 6 & 6y \end{pmatrix}$

At $(x,y) = (0,0)$, $D = \det\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} = -36$ saddle

At $(x,y) = (-2,-2)$, $D = \det\begin{pmatrix} -12 & 6 \\ 6 & -12 \end{pmatrix} = 108$

$\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow$ local maximum