More on extreme values and second partials

To find extrema of functions of more than one variable, we always look for the critical points first.

Definition. Let $z = f(x, y)$ be differentiable. If

$$\frac{\partial f}{\partial x}(a, b) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(a, b) = 0,$$

then (a, b) is a *critical point* of $f(x, y)$ and $c = f(a, b)$ is a *critical value*.

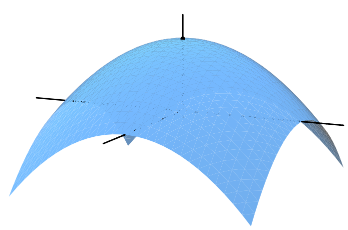
Note: The tangent plane to the graph $z = f(x, y)$ is horizontal at the critical point (a, b, c) because the normal vector

$$\mathbf{N} = \left[\frac{\partial f}{\partial x}(a, b)\right]\mathbf{i} + \left[\frac{\partial f}{\partial y}(a, b)\right]\mathbf{j} - \mathbf{k}$$

to the tangent plane reduces to $-\mathbf{k}$ at a critical point.

Example 1. Consider $f(x, y) = 9 - x^2 - y^2$.

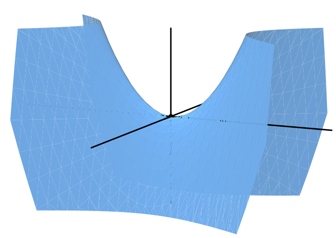
$$\begin{cases} \dfrac{\partial f}{\partial x} &= & -2x &= & 0 \\ \dfrac{\partial f}{\partial y} &= & -2y &= & 0 \end{cases}$$



yields only one critical point $(x, y) = (0, 0)$. This critical point corresponds to an absolute maximum of the $f(x, y)$.

Example 2. Consider $g(x, y) = y^2 - x^2$.

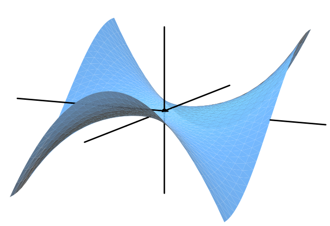
$$\begin{cases} \dfrac{\partial g}{\partial x} &= & -2x &= & 0 \\ \dfrac{\partial g}{\partial y} &= & 2y &= & 0 \end{cases}$$



yields only one critical point $(x, y) = (0, 0)$. This critical point corresponds to a saddle point in the graph of $g(x, y)$.

Example 3. Consider $h(x, y) = \frac{1}{3}x^3 - xy^2$.

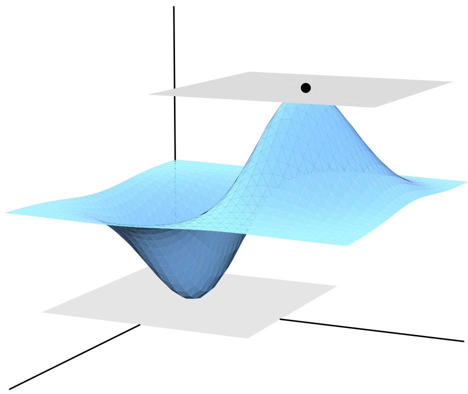
$$\begin{cases} \dfrac{\partial h}{\partial x} &= & x^2 - y^2 &= & 0 \\ \dfrac{\partial h}{\partial y} &= & -2xy &= & 0 \end{cases}$$

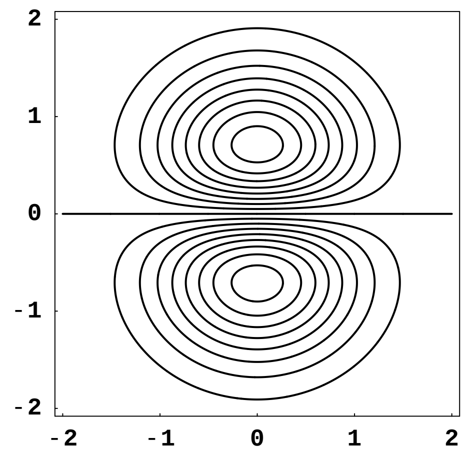


yields only one critical point $(x, y) = (0, 0)$. This critical point corresponds to a "monkey saddle" point in the graph of $h(x, y)$.

Theorem. If $c = f(a,b)$ is an extreme value, then (a,b) is a critical point of f.

Example. The function $f(x,y) = 3ye^{(-x^2-y^2)}$ has two critical points. One corresponds to a maximum, and one corresponds to a minimum.





How do we determine the type of a critical point?

Second Partials Test: Let (a,b) be a critical point of $f(x,y)$. Form the *Hessian* matrix

$$H(a,b) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

where $A = \dfrac{\partial^2 f}{\partial x^2}(a,b)$, $C = \dfrac{\partial^2 f}{\partial y^2}(a,b)$, and $B = \dfrac{\partial^2 f}{\partial x \partial y}(a,b) = \dfrac{\partial^2 f}{\partial y \partial x}(a,b)$. Let

$$D = \det H(a,b) = AC - B^2.$$

1. If $D > 0$, there are two cases.
   (a) If $A > 0$, then the critical point is a local minimum.
   (b) If $A < 0$, then the critical point is a local maximum.
2. If $D < 0$, then the critical point is a saddle.
3. If $D = 0$, the test is inconclusive.

Definition. If $D = 0$, we say that (a,b) is a *degenerate* critical point.

Let's return to the three examples discussed earlier.

Example 1. $f(x, y) = 9 - x^2 - y^2$

Example 2. $g(x, y) = y^2 - x^2$

Example 3. $h(x, y) = \frac{1}{3}x^3 - xy^2$

Warning. There are degenerate critical points that are extreme points. For example, the function

$$z = y^4 + x^2$$

has a degenerate critical point at $(0, 0)$, and it is an absolute minimum of the function.

Why does the Second Partials Test work the way that it does?

There is a proof of this test in Appendix E of our textbook, but I think that you would find the Discovery Project on p. 812 more informative. An important step in that project is step 4 where you use the technique of completing the square to understand the critical points of quadratic functions of the form

$$f(x, y) = ax^2 + bxy + cy^2$$

at $(0, 0)$.

When you look for extreme values of functions over bounded regions in the xy-plane, you need to analyze the critical points inside the region and analyze the behavior of the function along the boundary of the region. See pages 807 and 808 of your textbook for more details.

Multivariable chain rules

Today we also start our discussion of multivariable chain rules. There are two basic types, and both generalize the single variable chain rule.

Chain Rule—Type I

Consider a vector-valued function $\mathbf{P}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ which parametrizes a curve in the xy-plane and a function $f(x, y)$. How is the derivative df/dt related to the partial derivatives of $f(x, y)$ and the derivative $\mathbf{P}'(t)$?

Example. Consider $\mathbf{P}(t) = t\mathbf{i} + t^2\mathbf{j}$ and $f(x, y) = 2x^2 + y^2$. Let's calculate the derivative df/dt at $t = 1$.

The problem with this approach is that it ignores the fact that the function in question is really a composition of two functions. We can illustrate this composition symbolically by making a dependency chart.

Chain Rule. The derivative of the composition $f(\mathbf{P}(t))$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example. Back to the composition of $\mathbf{P}(t) = t\mathbf{i} + t^2\mathbf{j}$ and $f(x, y) = 2x^2 + y^2$. Let's calculate the derivative df/dt at $t = 1$ using the Chain Rule.