More on the multivariable chain rules

Chain Rule—Type I (continued)

Yesterday we learned that the multivariable chain rule for $f(\mathbf{P}(t))$ is:

**Chain Rule.** The derivative of the composition $f(\mathbf{P}(t))$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt}.$$

This version of the Chain Rule has an important formulation in terms of the gradient of $f$.

**Definition.** Given a function $f(x, y)$ that is differentiable at the point $(a, b)$. Then the *gradient vector* of $f$ at $(a, b)$ is the vector

$$\nabla f(a, b) = \frac{\partial f}{\partial x}(a, b)\,\mathbf{i} + \frac{\partial f}{\partial y}(a, b)\,\mathbf{j}.$$

Sometimes the gradient vector of $f$ is denoted $\mathbf{grad} f(a, b)$.

**Restatement of the Chain Rule.** The derivative of the composition $f(\mathbf{P}(t))$ is

$$\left.\frac{df}{dt}\right|_{t=t_0} = \nabla f(\mathbf{P}(t_0)) \cdot \mathbf{P}'(t_0).$$

**Example.** Once again we return to $\mathbf{P}(t) = t\mathbf{i} + t^2\mathbf{j}$ and $f(x, y) = 2x^2 + y^2$.

Animation of this chain rule

**Example.** Use the polar curve $r = \cos 2\theta$ to parametrize a curve $\mathbf{P}(t)$ in the $xy$-plane and consider the composition $f(\mathbf{P}(t))$ where

$$f(x, y) = y^2 - x^2.$$

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This chain rule has some important theoretical implications as well.

**Theorem.**

1. Let $f(x, y)$ be a differentiable function such that $\nabla f(x, y) = \mathbf{0}$ for all $(x, y)$. Then $f(x, y)$ is a constant function.

2. If $g(x, y)$ and $h(x, y)$ are two differentiable functions such that

$$\nabla g(x, y) = \nabla h(x, y)$$

for all $(x, y)$. Then $g(x, y) = h(x, y) + K$ for some constant $K$.

Chain Rule—Type II

For this situation, consider a function $f(x, y)$ of two variables and suppose that the variables $x$ and $y$ are functions of other variables.

For example, consider $x$ and $y$ as a function of the polar coordinates $r$ and $\theta$. That is,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

**Example.** Let $f(x, y) = xy + y^2$. What is the angular rate of change of $f(x, y)$ at the point $(x, y) = (1, 2)$?