Chain Rule—Type II

For this situation, consider a function $f(x, y)$ of two variables and suppose that the variables x and y are functions of other variables.

For example, consider x and y as a function of the polar coordinates r and θ. That is,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

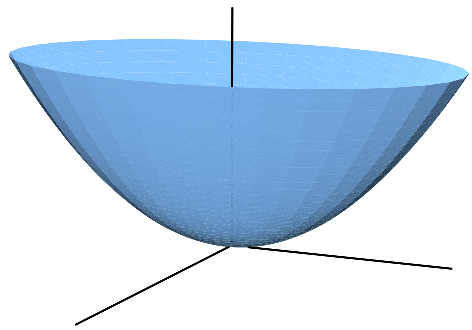
Example. Let $f(x, y) = xy + y^2$. What is the angular rate of change of $f(x, y)$ at the point $(x, y) = (1, 2)$?

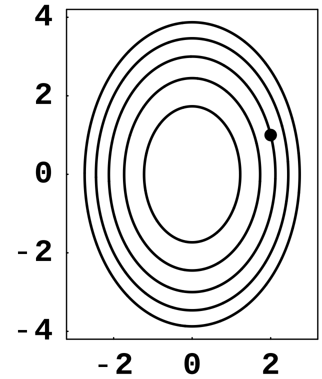
Directional derivatives

Partial derivatives only measure rates of change along paths parallel to the axes. Directional derivatives measure the rate of change of a function in any direction.

Example. On pages 788 and 790 of your textbook, there is a temperature map for parts of California and Nevada at 3:00 P.M. on a day in October. Let's estimate the rate that the temperature increases if we leave Reno traveling southeast (toward Las Vegas).

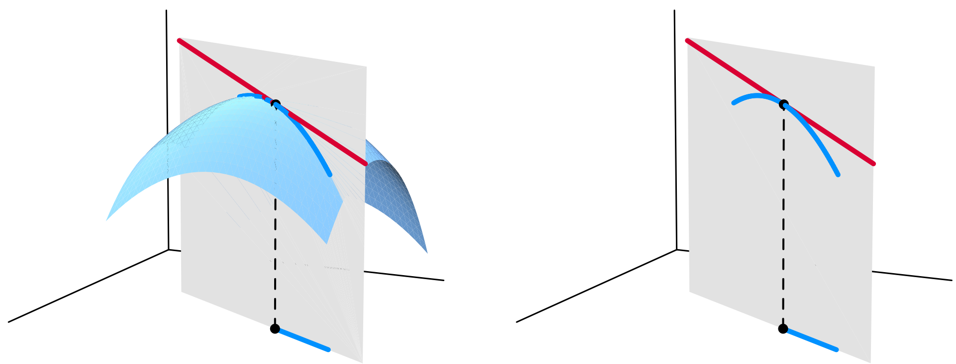
Example. Consider the function $f(x, y) = 2x^2 + y^2$. Here is its graph and level sets.





Let's calculate its derivative at the point $(2, 1)$ in the "northeast" direction.

On the web site there are links to two animations that illustrate the concept of a directional derivative.



Definition of a Directional Derivative. We start with the two-variable case. Define the "directional derivative of $f(x, y)$ at the point (a, b) in the $\mathbf{u}$-direction" by parametrizing the line through (a, b) using the direction vector $\mathbf{u}$. In other words, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, then the line is written in vector form as $\mathbf{L}(h) = (a\mathbf{i} + b\mathbf{j}) + h\mathbf{u}$ or in parametric form as $(x, y) = (a + u_1 h, b + u_2 h)$.

Then we compute

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(x, y) - f(a, b)}{h}.$$

Using vector notation with $\mathbf{P} = a\mathbf{i} + b\mathbf{j}$, the same limit is written as

$$D_{\mathbf{u}} f(\mathbf{P}) = \lim_{h \to 0} \frac{f(\mathbf{P} + h\mathbf{u}) - f(\mathbf{P})}{h}.$$

This vector notation generalizes nicely to functions of three variables or, in fact, to any number of variables.

Computing directional derivatives. A directional derivative for $f(x, y)$ at the point (a, b) can be computed by applying the Chain Rule to the composition $f(\mathbf{L}(h))$. Note that the vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is used only to indicate the direction, and consequently, it is *always a unit vector*. In other words, $u_1^2 + u_2^2 = 1$.

Theorem. $D_{\mathbf{u}} f(a, b) = [\nabla f(a, b)] \cdot \mathbf{u}$.

Example. Calculate the directional derivative of $f(x, y) = e^x \sin y$ at the point $(\ln 2, \pi/6)$ in the direction of $2\mathbf{i} + \mathbf{j}$.

This theorem tells us know how a function changes in any given direction, and in particular, it indicates directions of most rapid increase or decrease for the function. Since $\mathbf{u}$ is a unit vector,

$$\begin{aligned} D_{\mathbf{u}} f(a,b) &= (\nabla f(a,b)) \cdot \mathbf{u} \\ &= |\nabla f(a,b)|\,|\mathbf{u}|\,\cos\theta \\ &= |\nabla f(a,b)|\,\cos\theta, \end{aligned}$$

where θ is the angle between $\mathbf{u}$ and the gradient vector $\nabla f(a,b)$.

For what values of θ is this number largest? smallest? zero?

Theorem. The function $f(x,y)$ increases most rapidly in the direction of the gradient. The function is "constant" in directions perpendicular to the gradient.