Evaluating double integrals via iterated integration

Let's start with an example that we can compute using geometric methods.

Example. Consider

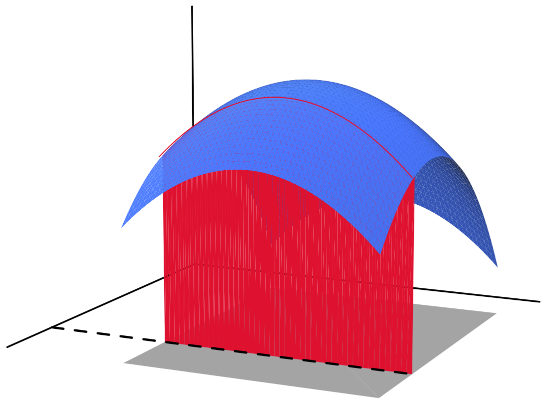
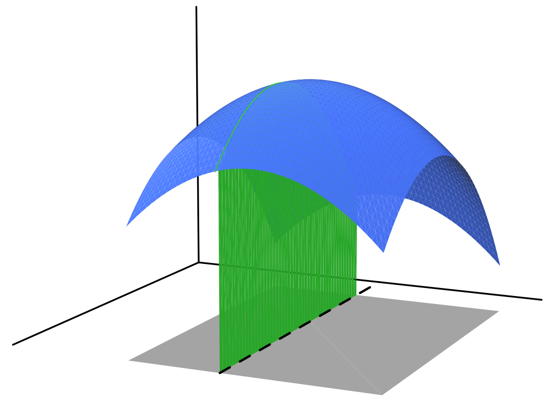
$$\iint_R \frac{y}{3}\, dA$$

where $R = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}$.

We can generalize this technique to obtain a method for calculating double integrals. Consider a positive function $f(x, y)$ and a rectangle

$$R = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$$

in the xy-plane. We can calculate the volume of the solid determined by R and $f(x, y)$ using x-slices or y-slices.

The basic idea is to compute the volume of the solid in question by integrating the areas of the slices. For example, suppose that we slice up the solid using y-slices. Let $A(y)$ denote the area of a y-slice. Then

$$\iint_R f(x,y)\,dA = \int_c^d A(y)\,dy.$$

Moreover, for any given y, the area of the y-slice is

$$A(y) = \int_a^b f(x,y)\,dx.$$

We obtain an *iterated integral* that yields the volume of the solid. That is,

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

We compute the inside integral treating y as a constant ("partial integration"), and then we compute the outside integral which only depends on y.

It is also possible to use x-slices rather than y-slices.

Theorem. (Fubini's Theorem) If $f(x,y)$ is continuous on the rectangle

$$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\},$$

then

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &= \int_c^d \int_a^b f(x,y)\,dx\,dy.\end{aligned}$$

Let's return to the example discussed earlier.

Example. Consider

$$\iint_R \frac{y}{3}\, dA$$

where $R = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}$.

The calculation that we did involved x-slices.

We can also calculate the integral using y-slices.

Example. Consider the double integral

$$\iint_R x\cos(xy)\, dA$$

where $R = \{(x,y)\,|\,0 \le x \le \pi/4,\ 0 \le y \le 2\}$.

Example. Calculate the average value of the function

$$f(x, y) = xe^{xy}$$

over the rectangle $R = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2\}$.