A little more about double integrals over general regions

I got a couple of questions after last class, and I think that everyone could benefit from hearing the answers.

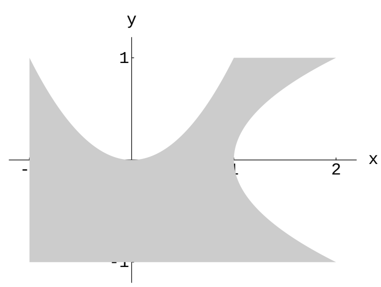
I briefly mentioned an example that is impossible to integrate using y-slices but is not difficult if one uses x-slices.

Example.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx$$

I also mentioned an old exam question:

Example. (Fall 2001 exam question) Consider the following region R in the xy-plane.



It is bounded by the curves $y = x^2$ and $x = y^2 + 1$ and the lines $y = -1$, $y = 1$, and $x = -1$. Calculate

$$\iint_R x\, dA.$$

I subdivided the region R into two subregions R_1 and R_2. The region R_1 was type 1, and the region R_2 was type 2. Furthermore,

$$\iint_R x\, dA = \iint_{R_1} x\, dA + \iint_{R_2} x\, dA.$$

You should do both integrals using the techniques discussed last class, but don't be surprised when you see that

$$\iint_{R_1} x\, dA = 0.$$

Using polar coordinates to calculate double integrals

Some important double integrals involve a lot of radial symmetry. Consequently, they are easier to evaluate if we use polar coordinates.

Two basic facts.

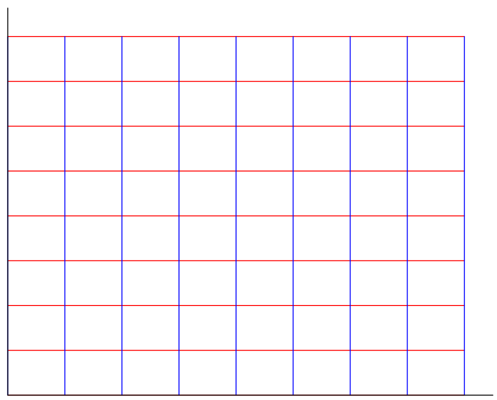
1. $\int_0^1 \sqrt{1-u^2}\,du = \frac{\pi}{4}$

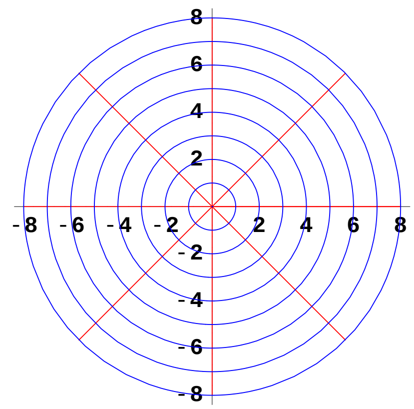
2. The volume of a solid sphere of radius 1 is $\frac{4}{3}\pi$.

Nonexample. Let's try to calculate the volume of a hemisphere of radius 1 using polar coordinates.

What's wrong with this calculation?

The problem with this calculation is that transforming to polar coordinates distorts area.





Recall the formula for area inside a polar curve (see the September 22 handout).

$$\text{area} = \int_{\theta_1}^{\theta_2} \frac{(r(\theta))^2}{2} \, d\theta.$$

Rewriting this formula as a double integral suggests the correct approach for converting double integrals to polar coordinates.

Differential form for area in terms of polar coordinates

$$dA = r\,dr\,d\theta = r\,d\theta\,dr$$

Example. Let's recalculate the volume of the hemisphere using this area adjustment factor.

Example. Calculate the volume of the region that is inside the sphere

$$x^2 + y^2 + z^2 = 4$$

and above the plane $z = 1$.