Converting integrals to spherical coordinates

Spherical coordinates can often be used to simplify triple integrals that possess spherical symmetry.

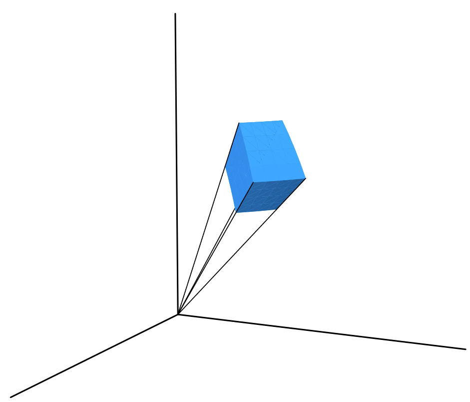
The relationship between the spherical coordinates (ρ, θ, ϕ) of a point and its rectangular coordinates (x, y, z) is best remembered using the equations

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\,\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\,\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

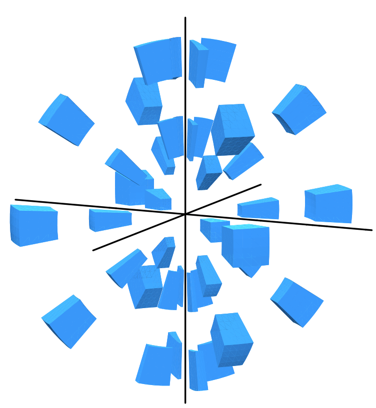
because $r = \rho\sin\phi$. Also, $x^2 + y^2 + z^2 = \rho^2$. Recall that we tend to follow the conventions that $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$.

Whenever we use a new coordinate system to do a triple integral, we need a volume adjustment factor. In other words, we need to determine how the volume of a "cube" in the new coordinates converts to a volume in rectangular coordinates.

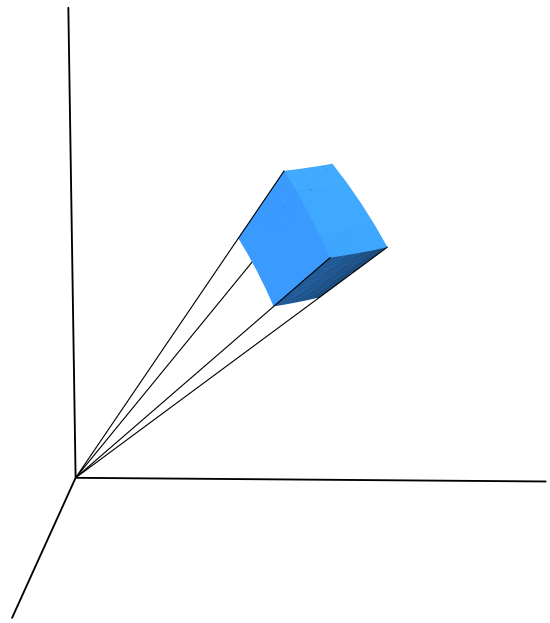
Consider a "spherical cube" with sides $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$.



If we look at various cubes all with the same $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$, we see that their volumes vary greatly.



Let's see if we can approximate this variation.



In summary, our volume conversion factor for spherical coordinates is

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example. Find the amount of ice creme in a spherical cone determined by the inequalities

$$x^2 + y^2 + z^2 \leq 16, \quad z^2 \geq 3(x^2 + y^2), \quad \text{and} \quad z \geq 0.$$

Example. Find the volume of the "apple" determined by the spherical equation

$$\rho = 1 + \sin\phi.$$