More on Green's Theorem

Green's Theorem relates line integrals of vector fields in the xy-plane to double integrals.

Theorem. (Green's Theorem) Let C be a positively-oriented, simple, closed curve in the plane and let D denote the region it encloses. Then

$$\oint_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Example. Let C be the perimeter of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Calculate

$$\oint_C x\,dx + xy\,dy.$$

Note: If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ has a potential function, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0,$$

and we see that $\oint_C P\,dx + Q\,dy = 0$.

Example. Compute the line integral

$$\oint -y^3\,dx + x^3\,dy$$

over the unit circle in the positively-oriented direction.

The curl of a planar vector field

I would like to use Green's Theorem to explain one of the basic concepts in vector analysis—the curl of a vector field—in the case where the vector field $\mathbf{F}$ is a planar vector field. It helps if you consider the vector field $\mathbf{F}$ as a velocity field of a fluid and you imagine a little "paddle wheel" suspended in the fluid (see Figure 6 on p. 963 of your text). We would like to measure how much the paddle wheel rotates as it moves through the fluid.

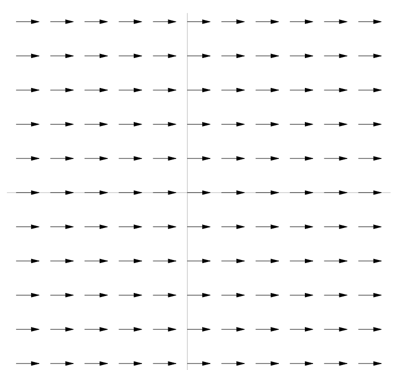
For velocity fields of fluids, the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is called the *circulation* of the fluid along the curve. The angular velocity of the paddle wheel at the point (x_0, y_0) is one-half of the *circulation density* of the velocity field at the point (x_0, y_0). Circulation density is defined to be the limit

$$\lim_{r\to 0} \frac{\int_C \mathbf{F}\cdot\mathbf{T}\,ds}{\text{area inside } C}$$

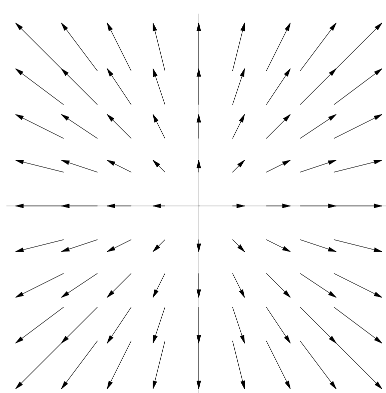
where C is a circle of radius r centered at (x_0, y_0).

Here are five examples to illustrate this relationship.

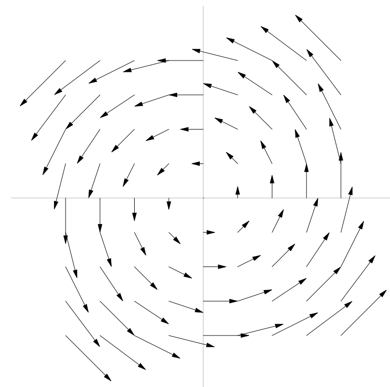
Example 1. Let $\mathbf{F}(x, y) = \mathbf{i}$.



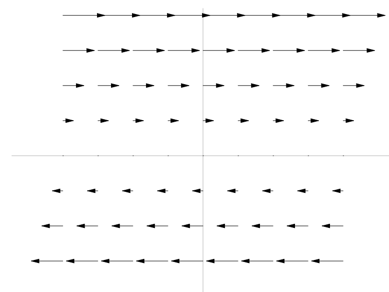
Example 2. Let $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.



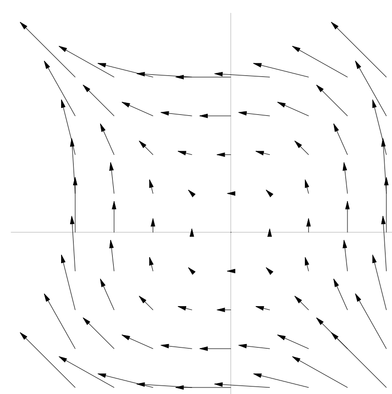
Example 3. Let $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.



Example 4. Let $\mathbf{F}(x, y) = y\mathbf{i}$.



Example 5. Let $\mathbf{F}(x, y) = -y^2\mathbf{i} + x^2\mathbf{j}$.



Definition. For a planar vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, the curl of $\mathbf{F}(x, y)$ is the vector field

$$\operatorname{curl} \mathbf{F}(x, y) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

To interpret the curl of $\mathbf{F}$ in this situation, we use Green's Theorem.

Theorem. (The vector form of Green's Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA.$$

How does this help us interpret the curl of $\mathbf{F}$?