Flux integrals

Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

and a surface S, the flux of $\mathbf{F}$ across S is the surface integral

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS,$$

where $\mathbf{n}$ is the unit normal vector at each point of S.

Special case: Again we assume that the surface S is the graph of a function $g(x, y)$ over a region R in the xy-plane. Note that

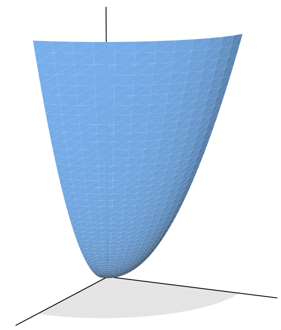
$$\mathbf{n} = \pm\frac{\mathbf{N}}{|\mathbf{N}|} \text{ where } \mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}.$$

In this case, we compute the flux of $\mathbf{F}$ across S by

$$\begin{aligned}\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS &= \pm\iint_R \left(\mathbf{F}\cdot\frac{\mathbf{N}}{|\mathbf{N}|}\right) |\mathbf{N}|\, dA \\ &= \pm\iint_R (\mathbf{F}\cdot\mathbf{N})\ dA.\end{aligned}$$

Example. Let S be the part of the circular paraboloid $z = 2x^2 + 2y^2$ that is in the first octant and below the plane $z = 6$. Let's calculate the flux across S in the direction of the outer normal for the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$



Summary. A complete discussion of vector analysis involves two more theorems that are both similar to Green's Theorem.

Theorem. (Stokes's Theorem) Let S be an oriented surface that is bounded by a simple, closed curve C with positive orientation. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS.$$

Stokes's Theorem is used to explain why curl measures the local spinning caused by the vector field. (You should think of the little paddle wheel.)

Theorem. (Divergence Theorem) Let R be a solid region in space bounded by the surface S. Orient S using the outward pointing unit normal vector $\mathbf{n}$. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_R \operatorname{div} \mathbf{F} \, dV.$$

The Divergence Theorem is using to explain why divergence measures the local rate at which the streamlines of the vector field are expanding.

These two important theorems are discussed in Sections 13.7 and 13.8 of the textbook (pp. 959–972), and all of the major theorems of vector analysis are summarized on page 973 of your text.