Using the Linearity Principle, we can produce many solutions from just a few:

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is a solution for any choice of constants k_1 and k_2.

Now recall the example we did last class:

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

We can use this observation to solve any initial-value problem involving this system.

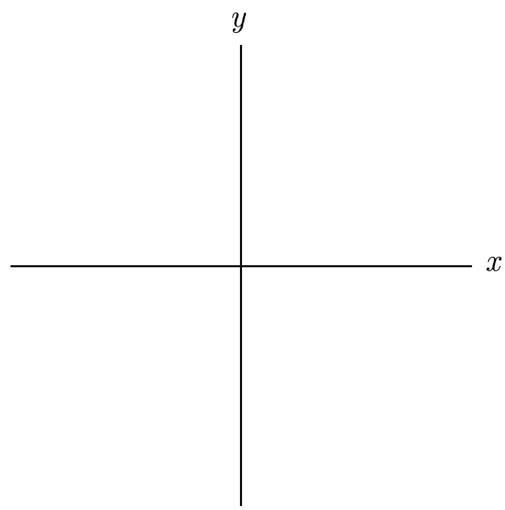
Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

In general, how many solutions do we need to be able to solve any initial-value problem?

How do we find a few solutions to get started?

Let's return to our example and see what is so special about $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.



We want nonzero initial conditions $\mathbf{Y}_0$ (vectors) so that

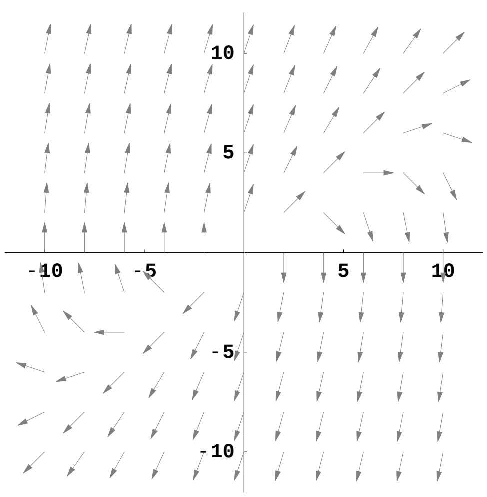
$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

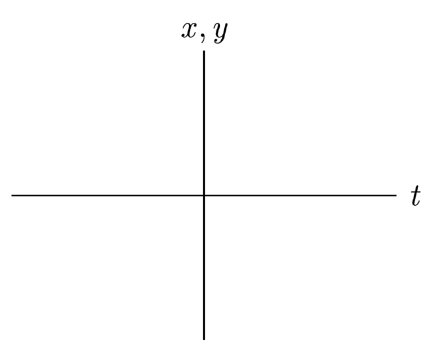
for some scalar λ.

Terminology: The scalar λ is called an *eigenvalue* of the matrix $\mathbf{A}$ and the vector $\mathbf{Y}_0$ is called an *eigenvector* associated to the eigenvalue λ.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{Y}.$$





"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$