Tacoma Narrows Bridge

In this project, we discuss a model of the Tacoma Narrows Bridge. This model is based on the observation that while stretched cables can be reasonably modeled as springs, contracted (or slack) cables do not exert a restoring force. This one-sided Hooke's Law may remind you of Project 1 where we considered a mass attached to a spring and a rubber band. In this project, we extend the study of the system in Project 1 by adding periodic forcing. This model displays very interesting (and scary) behavior, particularly when the solutions are interpreted in terms of the behavior of a suspension bridge. Since this nonlinear model displays nontrivial behavior, it is reasonable to suspect that more complicated models could display the same complicated solutions.

In this project we compare the usual forced harmonic oscillator

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + k_1 y = 10 + A\sin\omega t$$

to the forced mass-spring system with a rubber band added. The equation for the forced system is

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + k_1 y + k_2 h(y) = 10 + A\sin\omega t,$$

where $h(y)$ is defined piecewise as

$$h(y) = \begin{cases} y, & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Recall from Project 1 that the term $k_1 y$ is due to the force from the spring, $b(dy/dt)$ represents damping, the 10 on the right-hand side is (a rough approximation to) the constant force of gravity, and $k_2 h(y)$ is the force of a rubber band, which is a restoring force proportional to the displacement when stretched but no force when contracted. The new term, $A\sin\omega t$, represents periodic external forcing.

We first review the behavior of the standard forced harmonic oscillator (no rubber band present), particularly as the parameters b and ω vary. Then we compare these solutions to the solutions of the forced mass-spring system with rubber band for some specific parameter values. The numerics of this project are delicate and require patience.

Using $A = 0.1$ throughout the project, address the following items in your report:

1. (Undamped, forced harmonic oscillator) First consider solutions to

$$\frac{d^2y}{dt^2} + k_1 y = 10 + A\sin\omega t.$$

   Using the Method of the Lucky Guess, determine a particular solution to the equation. Then using a numerical solver or the formulas for the solutions, estimate the amplitudes of the solutions for frequencies ω in the interval $0 \leq \omega \leq 5$.

2. (Damped, forced oscillator) Assume that forcing is present along with some damping ($b > 0$). That is, consider the equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + k_1 y = 10 + A\sin\omega t.$$

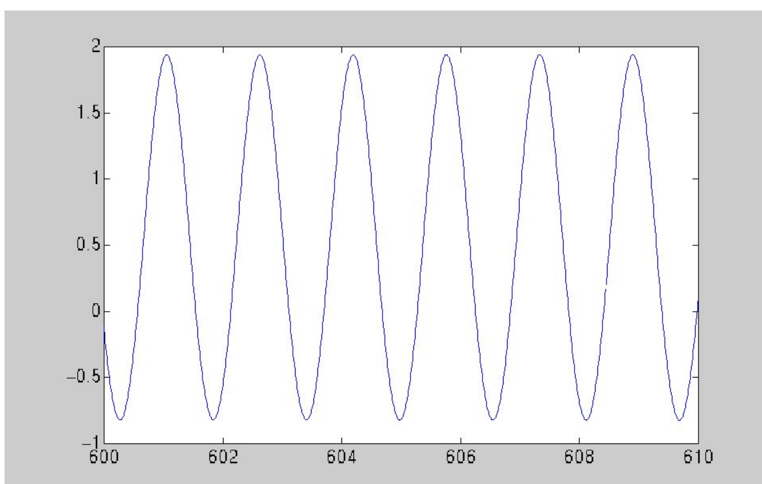
How long does it take any given solution to get close to the steady-state solution? Using a numerical solver or the formulas for the solutions, estimate the maximum amplitudes of the solutions for frequencies ω in the interval $0 \le \omega \le 5$. Graph the maximum amplitude as a function of frequency for different values of b, all on the same set of axes. What happens to these graphs as $b \to 0$?

3. (Small-amplitude periodic solutions for the forced mass-spring system with a rubber band) Fix $b = 0.01$ and $\omega = 4$, and use analytic techniques to calculate a periodic solution with small amplitude for the equation

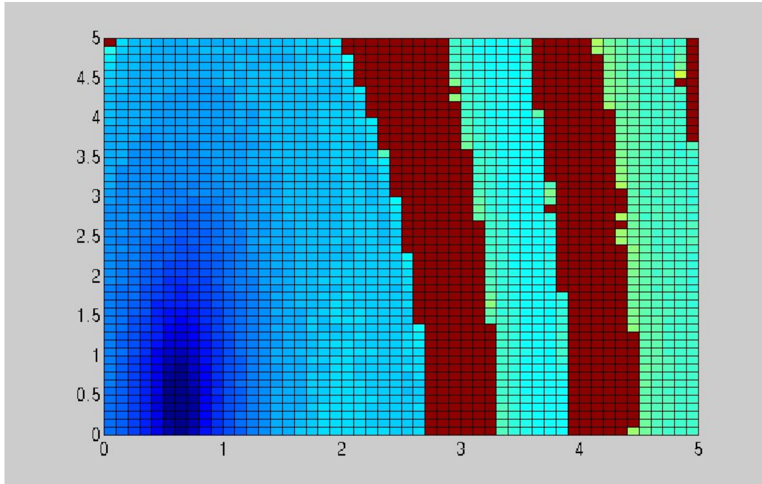
$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + k_1 y + k_2 h(y) = 10 + A\sin\omega t.$$

Then verify your calculation using a numerical solver. Graph the solution $y(t)$ and its corresponding solution curve in the phase plane. What are the initial conditions for this solution?

4. (Large-amplitude periodic solutions for the forced mass-spring system with a rubber band) Surprisingly, the equation in Part 3 has a second periodic solution with larger amplitude that "attracts" many other solutions just as the small amplitude periodic solution attracts many solutions. The following graph is the graph of the large amplitude periodic solution for values of k_1 and k_2 that are close to but not the same as yours.



Using the same values of b and ω as in Part 3, search for initial conditions that lead to this large amplitude solution. This is delicate work. You will need to try many different initial conditions and allow the numerical method to work for a long time interval (on the order of $0 \le t \le 1000$). Then check to see if the solution has approached a periodic solution. As an aid, we include the following figure.



This is a picture of a quadrant of the yv-plane. Solutions with initial conditions in the blue bands approach the small-amplitude periodic orbit. Solutions with initial conditions in the red bands do not approach the small amplitude solution for $0 \leq t \leq 1000$. This figure was produced using $k_1 = 13$ and $k_2 = 4$, but it should give you some idea of how to start working for your values on k_1 and k_2.