1. (18 points) Short answer questions: The answers to these questions need only consist of one or two sentences, but you do need to show enough work so that we can tell that you are not just guessing. Partial credit will be awarded only in exceptional situations.

    (a) Find all equilibrium solutions of the differential equation

    $$\frac{dy}{dt} = \frac{(t^2 - 4)(1 + y)(y^2 - 2)e^y}{(t - 1)(3 - y)}.$$

    Want $\frac{dy}{dt} = 0$ for all t.

    $\frac{dy}{dt} = 0$ for all $t \iff 1 + y = 0$ or $y^2 - 2 = 0$

    $\iff y = -1, \pm\sqrt{2}$

    (b) Is $y(t) = t^2$ a solution to the differential equation $\frac{dy}{dt} = y^2 - t^2 y + y - t^2 + 2t$? Why?

    If $y(t) = t^2$, $\frac{dy}{dt} = 2t$ and

    $y^2 - t^2 y^2 + y - t^2 + 2t = t^4 - t^4 + t^2 - t^2 + 2t$

    $= 2t$ ✓

    Yes, $y(t) = t^2$ is a solution.

    (c) Give an example of a first-order, autonomous, linear, nonhomogeneous differential equation that has the equilibrium solution $y(t) = 3$ for all t.

    $\frac{dy}{dt} = y - 3$ autonomous

                          linear

                          nonhomogeneous

    $\frac{dy}{dt} = 0$ for all t if $y = 3$.

2. (20 points) Consider the following 8 first-order equations:

1. $\dfrac{dy}{dt} = y^2 + y - 2$ 2. $\dfrac{dy}{dt} = y^2 - y - 2$ 3. $\dfrac{dy}{dt} = t^2 - 2$ 4. $\dfrac{dy}{dt} = t^2 - 2t$

5. $\dfrac{dy}{dt} = ty - t$ 6. $\dfrac{dy}{dt} = ty + t$ 7. $\dfrac{dy}{dt} = y + 2t$ 8. $\dfrac{dy}{dt} = y - 2t$

$t(y-1)$ $t(y+1)$

Four of the associated slope fields are shown on the next page. Pair the slope fields with their associated equations. Provide a brief justification for your choice. **You will not receive any credit unless you justify your selection.**

(a) The equation for slope field A is ___1___. My reason for choosing this answer is:

Slope field is autonomous $\Rightarrow$ #1, 2

$(y^2 + y - 2) = (y+2)(y-1)$ $\leftarrow$ eq. solns

$(y^2 - y - 2) = (y-2)(y+1)$ $y = -2$ and $y = 1$

(b) The equation for slope field B is ___5___. My reason for choosing this answer is:

Slope field is non autonomous and not

$\frac{dy}{dt} = f(t) \Rightarrow$ # 5, 6, 7, 8. Eq solution

$y = 1 \Rightarrow$ # 5.

(c) The equation for slope field C is ___7___. My reason for choosing this answer is:

Slope field is nonautonomous and not

$\frac{dy}{dt} = f(t)$. No eq. solns $\Rightarrow$ #7, 8.

Along t-axis, slopes positive if $t > 0 \Rightarrow$ #7

(d) The equation for slope field D is ___4___. My reason for choosing this answer is:

Slope field from equation of form

$\frac{dy}{dt} = f(t). \Rightarrow$ #3, 4. $\frac{dy}{dt} = 0$ along
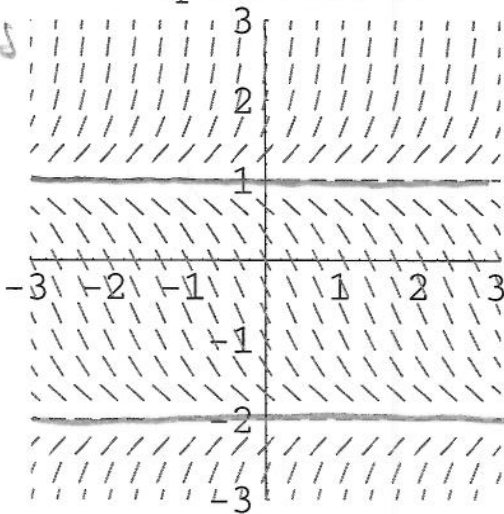
y-axis $\Rightarrow$ #4

2. (continued) **Answer this question on the previous page.** The equations are provided here for your convenience:

1. $\dfrac{dy}{dt} = y^2 + y - 2$ 2. $\dfrac{dy}{dt} = y^2 - y - 2$ 3. $\dfrac{dy}{dt} = t^2 - 2$ 4. $\dfrac{dy}{dt} = t^2 - 2t$

5. $\dfrac{dy}{dt} = ty - t$ 6. $\dfrac{dy}{dt} = ty + t$ 7. $\dfrac{dy}{dt} = y + 2t$ 8. $\dfrac{dy}{dt} = y - 2t$
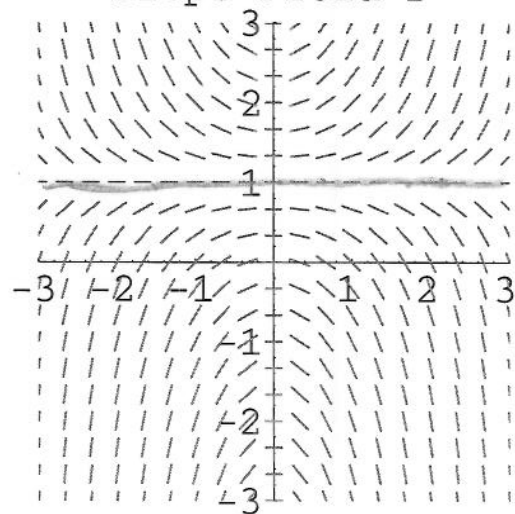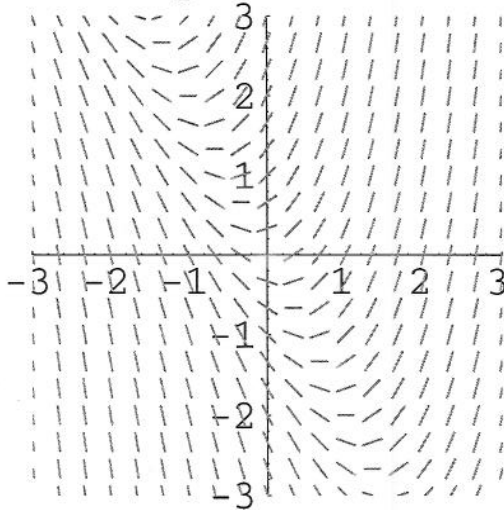
Slope Field A   Slope Field B

autonomous

eq.
colns

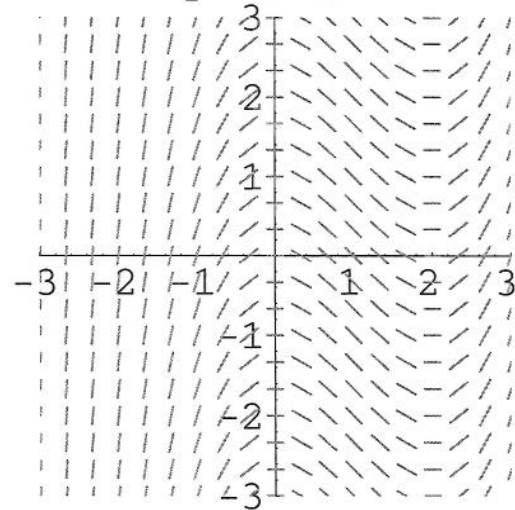## Slope Field C   Slope Field D

no
eq.
solns

← eq.
soln

$\dfrac{dy}{dt} = f(t)$

3. (19 points) Calculate the general solution of the differential equation

$$\frac{dy}{dt} = 2y^2 - 4ty^2.$$

Separate variables:

$$\frac{dy}{dt} = (2-4t)y^2$$

$$\int y^{-2} \, dy = \int 2-4t \, dt$$

$$-y^{-1} = 2t - 2t^2 + c$$

$$y^{-1} = 2t^2 - 2t + k$$

$$y = \frac{1}{2t^2 - 2t + k}$$

k arbitrary constant.

Also, we need to include the equilibrium solution $y(t) = 0$ for all t to solve all initial-value problems.

4. (20 points) Consider the initial-value problem

$$\frac{dy}{dt} = 3 - y^2, \quad y(0) = 0.$$

(a) Calculate the results of Euler's method applied to this initial-value problem on the interval $[0, 2]$ with 4 subdivisions. (Make sure that you show enough calculations so that we can see that you know the method.) Then graph your results. Make sure that you label the axes on your graph and clearly indicate the scale on each axis. You can use a calculator and do all calculations to 2 decimal places if you wish.

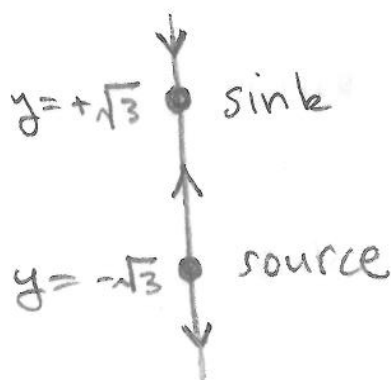| t_k	y_k	$f(t_k, y_k)$
0	0	3
.5	1.5	.75
1	1.87	-.5
1.5	1.62	.38
2.0	1.81	
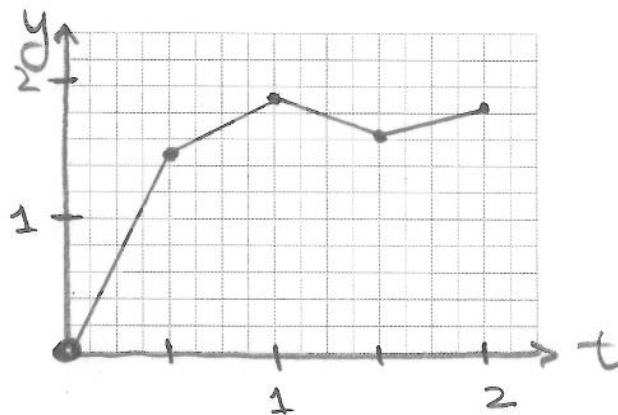
$$\Delta t = \frac{2-0}{4} = \frac{1}{2} = .5$$

$$y_{k+1} = y_k + f(t_k, y_k)(.5)$$

(b) Sketch the phase line for this differential equation. What does it tell you about your results in part (a)?

eq. points at $y = \pm\sqrt{3} \approx \pm 1.73$

$y = +\sqrt{3}$ • sink

$y = -\sqrt{3}$ • source

The actual solution is increasing and tends to $\sqrt{3}$ as $t \to \infty$. The oscillations in the approximate solution are due to numerical error.

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5. (20 points) The air in a small rectangular room 20 ft by 5 ft by 10 ft is 4% carbon monoxide. Starting at $t = 0$, air containing 1% carbon monoxide is blown into the room at the rate of 100 ft^3/min and well mixed air flows out through a vent at the same rate.

$1000 \ ft^3$

(a) Write an initial-value problem that describes the amount of carbon monoxide in the room.

C = amount of carbon monoxide
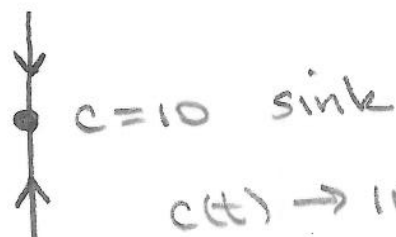
t = time (min)

$$\frac{dc}{dt} = 1 - \frac{c}{10}$$

$$c(0) = 40$$

(b) Sketch the phase line corresponding to the initial-value problem in part (a), and determine how much carbon monoxide will be in the room over the long term.

eq. solns

$$1 - \frac{c}{10} = 0$$

$$\Rightarrow c = 10$$

$c = 10$ sink

$c(t) \rightarrow 10$ as $t \rightarrow \infty$

(c) When will the air in the room be 2% carbon monoxide?

The equation is linear (NH) and separable. Many methods work. We use lucky guess. The eq. solution $c = 10$ is our particular solution. Gen soln of AHE is $ke^{-t/10}$. $\Rightarrow c(t) = 10 + ke^{-t/10}$

$c(0) = 40 \Rightarrow k = 30$.

$c(t) = 10 + 30 e^{-t/10}$. Want t such that

$c(t) = 20 \Rightarrow 30 e^{-t/10} = 10$

$e^{-t/10} = 1/3$

$-t/10 = -\ln 3 \Rightarrow t = 10 \ln 3$

≈ 10.98 min

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