

1. (16 points) Note that parts c and d of this problem are on the next page.
Consider the second-order equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -5 \cos 5t.$$

- (a) What can you say about the long-term behavior of solutions without solving for the general solution? Be as specific as possible.

All solutions tend to the steady-state solution, and the angular frequency of the steady-state solution is 5.

- (b) Determine a particular solution to this differential equation.

Complexify: $\frac{d^2y_c}{dt^2} + 4\frac{dy_c}{dt} + 20y_c = -5e^{(5i)t}$

Guess $y_c(t) = ae^{(5i)t}$. Then

$$\begin{aligned} \frac{d^2y_c}{dt^2} + 4\frac{dy_c}{dt} + 20y_c &= a(-25+20i+20)e^{(5i)t} \\ &= a(-5+20i)e^{(5i)t} \stackrel{(5i)t}{=} -5e^{(5i)t} \end{aligned}$$

$$\Rightarrow a = \frac{-5}{-5+20i} = \frac{1}{1-4i} \left(\frac{1+4i}{1+4i} \right) = \frac{1+4i}{17}$$

$$y_p = \operatorname{Re}(y_c) = \operatorname{Re}\left(\frac{1+4i}{17}(\cos 5t + i \sin 5t)\right)$$

$$\Rightarrow y_p(t) = \frac{1}{17}(\cos 5t - 4 \sin 5t)$$

1. (continued) Here is the differential equation from the previous page:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -5 \cos 5t.$$

- (c) Find the general solution to this differential equation.

$$\text{char poly } \lambda^2 + 4\lambda + 20 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-80}}{2} \\ = -2 \pm 4i$$

General solution:

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t \\ + \frac{1}{17} (\cos 5t - 4 \sin 5t)$$

- (d) What can you say about the long-term behavior of the solutions given your results from parts b and c? Be as specific as possible.

The amplitude of the steady-state solution is $|a| = \frac{1}{17} \sqrt{1+16} = \frac{1}{\sqrt{17}}$.

The phase angle of the steady-state is determined by $\arctan 4$. In degrees it is approx 104° .

The exponential rate of decay to the steady-state is determined by the real part of the eigenvalue, which is -2 .

2. (15 points) Solve the initial-value problem

$$\frac{dx}{dt} = 4x - 2y$$

$$\frac{dy}{dt} = x + y$$

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

with $(x(0), y(0)) = (-1, -2)$.

$$\lambda^2 - 5\lambda + 6 = 0 \\ (\lambda - 2)(\lambda - 3)$$

$$\lambda = 2 \text{ eigenvectors: } \begin{cases} 4x_0 - 2y_0 = 2x_0 \\ x_0 + y_0 = 2y_0 \end{cases} \Rightarrow x_0 = y_0$$

$$\lambda = 3 \text{ eigenvectors: } \begin{cases} 4x_0 - 2y_0 = 3x_0 \\ x_0 + y_0 = 3y_0 \end{cases} \Rightarrow x_0 = 2y_0$$

General solution:

$$\mathbf{Y}(t) = k_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

To solve the IVP, we evaluate at $t=0$

$$\begin{cases} k_1 + 2k_2 = -1 \\ k_1 + k_2 = -2 \end{cases} \Rightarrow \begin{aligned} k_2 &= 1 \\ k_1 &= -3 \end{aligned}$$

The desired solution is

$$\mathbf{Y}(t) = -3e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^{2t} + 2e^{3t} \\ -3e^{2t} + e^{3t} \end{pmatrix}$$

3. (15 points) Note that part c of this problem is on the next page.

(a) Calculate $\mathcal{L}^{-1} \left[\frac{2s+5}{s^2+2s+3} \right]$. $s^2+2s+3 = (s+1)^2 + 2$

$$\frac{2s+5}{(s+1)^2+2} = \frac{2(s+1)}{(s+1)^2+2} + \frac{3}{(s+1)^2+2}$$

$$\Rightarrow \mathcal{L}^{-1} \left[\frac{2s+5}{(s+1)^2+2} \right] = 2e^{-t} \cos \sqrt{2}t + \frac{3}{\sqrt{2}} e^{-t} \sin \sqrt{2}t.$$

(b) Calculate the Laplace transform $\mathcal{L}[y]$ for the solution $y(t)$ to the initial-value problem

$$\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 3y = \delta_4(t), \quad y(0) = 2, \quad y'(0) = 1.$$

DO NOT CALCULATE A FORMULA FOR $y(t)$ HERE.

$$(s^2 \mathcal{L}[y] - 2s - 1) + 2(s \mathcal{L}[y] - 2) + 3 \mathcal{L}[y] = e^{-4s}$$

$$(s^2 + 2s + 3) \mathcal{L}[y] = 2s + 5 + e^{-4s}$$

$$\mathcal{L}[y] = \frac{2s+5}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3}$$

3. (continued)

(c) Calculate the solution $y(t)$ to the initial-value problem in part b.

We need $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+2s+3}\right]$ and

consequently $\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+3}\right] =$

$$\mathcal{L}^{-1}\left[\frac{1}{2\sqrt{2}}\left(\frac{t^2}{(s+1)^2+2}\right)\right] = \frac{1}{2\sqrt{2}} e^{-t} \sin \sqrt{2}t$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+2s+3}\right] = \frac{u_4(t)}{\sqrt{2}} e^{-(t-4)} \sin \sqrt{2}(t-4)$$

\Rightarrow

$$y(t) = 2e^{-t} \cos \sqrt{2}t + \frac{3}{2\sqrt{2}} e^{-t} \sin \sqrt{2}t$$

$$+ \frac{u_4(t)}{\sqrt{2}} e^{-(t-4)} \sin \sqrt{2}(t-4).$$

4. (12 points) Short answer questions: The answers to these questions need only consist of one or two sentences. Partial credit will be awarded only in exceptional situations.

- (a) Find one solution of the differential equation $dy/dt = ty + 3y - 2t - 6$.

$$\frac{dy}{dt} = (t+3)y - 2(t+3) = (t+3)(y-2)$$

The equilibrium solution $y(t) = 2$ for all t is a solution.

- (b) Sketch the solution curve for the initial-value problem $dx/dt = -2x$, $dy/dt = -2y$, and $(x_0, y_0) = (2, 1)$.

The vector field $F(Y) = -2Y$.

All solutions are straight-line solutions.



- (c) Find all equilibrium solutions of the equation $d^2y/dt^2 + 4y = \sin 2t$.

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -4y + \sin 2t \end{cases} \Rightarrow \begin{aligned} v &= 0 \\ 4y &= \sin t \end{aligned}$$

These two equations contradict one another \Rightarrow no equilibrium solutions.

5. (12 points) Are the following statements true or false? You must justify your answers to receive any credit.

- (a) Every solution of $dy/dt = y + e^{-t}$ tends either to $+\infty$ or to $-\infty$ as $t \rightarrow \infty$.

False. This is a nonhomogeneous linear equation. Guess $y_p(t) = ae^{-t}$
 $\Rightarrow \frac{dy_p}{dt} = -ae^{-t} \stackrel{?}{=} ae^{-t} + e^{-t} \Rightarrow -a = a+1 \Rightarrow a = -\frac{1}{2}$

General solution: $y(t) = ket^{-t} - \frac{1}{2}e^{-t}$.
If $k=0 \Rightarrow y(t) = -\frac{1}{2}e^{-t} \rightarrow 0$ as $t \rightarrow \infty$.

- (b) The function $\mathbf{Y}(t) = (\cos 2t, \sin t)$ is not a solution to any linear system.

True. If $\lambda + i\beta$ is an eigenvalue, then the solution contains terms of the form $e^{\lambda t} \cos \beta t$ and $e^{\lambda t} \sin \beta t$. The first component $\Rightarrow \beta = 2$ and the second component $\Rightarrow \beta = 1$.

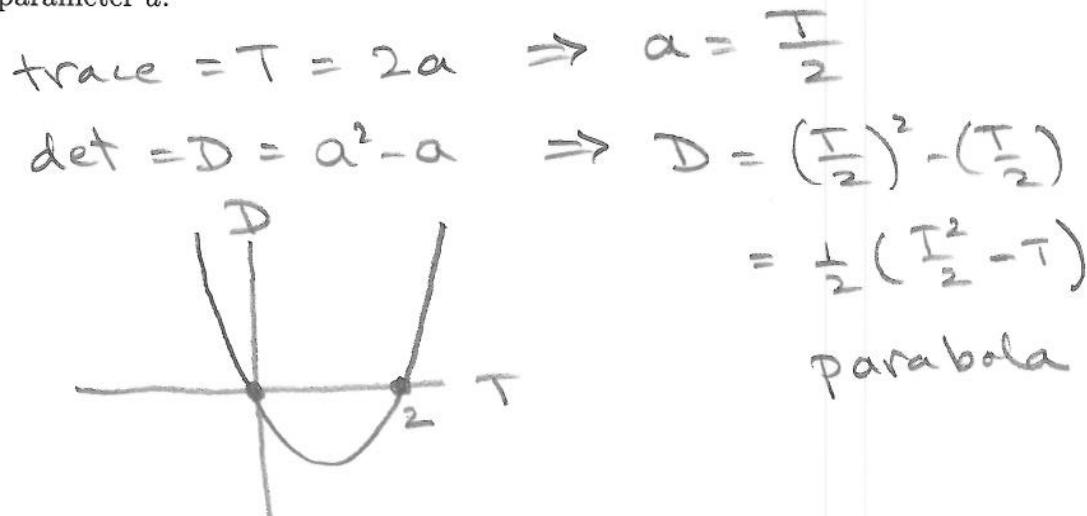
- (c) If the function $(x_1(t), y_1(t)) = (\cos t, \sin t)$ is a solution of a first-order autonomous system, then the function $(x_2(t), y_2(t)) = (-\sin t, \cos t)$ is also a solution of the same system.

True. Note that $\cos(t + \frac{\pi}{2}) = -\sin t$ and $\sin(t + \frac{\pi}{2}) = \cos t$. Since the system is autonomous, we know that $(\cos(t + \frac{\pi}{2}), \sin(t + \frac{\pi}{2}))$ is a solution.

6. (15 points) Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix} \mathbf{Y}.$$

- (a) Sketch the curve in the trace-determinant plane that is obtained by varying the parameter a .



- (b) Determine all bifurcation values of a and briefly discuss the different types of phase portraits that are exhibited in this one-parameter family.

Determine where this curve crosses the parabola $D = \frac{T^2}{4}$: $\frac{T^2}{4} = \frac{1}{4}(T^2 - T)$

$$\frac{T^2}{2} = \frac{T^2}{2} - T$$

$$\Rightarrow T = 0$$

Also where does this curve cross the T -axis : $T=0$ and $\frac{T}{2}-1=0$

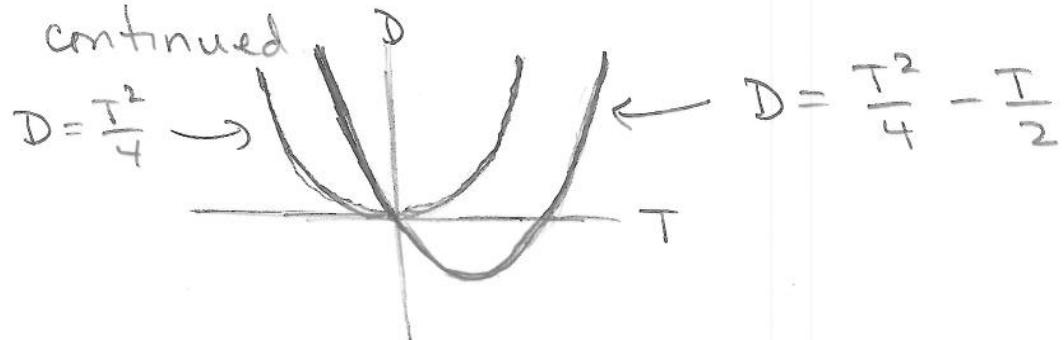
$T=\frac{1}{2}$ and $T=2$

MA 226

Find A

May 7, 2009

6(b) continued



Bifurcation values: $T = 0$ and $T = 2$

$\Rightarrow \alpha = 0$ and $\alpha = 1$

$\alpha < 0 \Rightarrow T < 0 \Rightarrow$ spiral sinks.

$0 < \alpha < 1 \Rightarrow D < 0 \Rightarrow$ saddles

$1 < \alpha \Rightarrow T > 2$ to the right
of the critical parabola

\Rightarrow real sources.

7. (15 points) A can of soda just removed from the refrigerator is 50° . After 10 minutes of sitting in a 70° room, its temperature is 54° .

- (a) Assume that Newton's Law of "Cooling" applies: The rate of change of temperature is proportional to the difference between the current temperature and the ambient temperature. Write an initial-value problem that models the temperature of the can of soda.

$$\frac{dT}{dt} = k(70 - T)$$

$$\begin{aligned} T &= \text{temperature (dgs)} \\ t &= \text{time (min)} \\ T(0) &= 50^\circ \end{aligned}$$

where k is a proportionality constant (see part c)

- (b) Sketch the phase line corresponding to the initial-value problem in part (a), and determine the temperature of the can of soda over the long term.



As $t \rightarrow \infty$, $T \rightarrow 70^\circ$

for all initial conditions.

- (c) How long does it take for the can of soda to reach 60° ?

First, we need to determine k .

The differential equation is separable and linear. We use the Extended

Linearity principle. AHE is $\frac{dT}{dt} = -kT$

Equilibrium solution $T(t) = 70$ is one particular solutn. \Rightarrow

General solution: $T(t) = C e^{-kt} + 70$

Since $T(0) = 50 \Rightarrow T(t) = 70 - 20 e^{-kt}$

MA 226

Final A

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#7(c) continued.

$$T(10) = 54 \Rightarrow 54 = 70 - 20e^{-10k}$$

$$20e^{-10k} = 16$$

$$e^{-10k} = \frac{16}{20} = .8$$

$$-10k = \ln .8$$

$$k = .0223$$

To find t such that $T(t) = 60$,
we solve

$$60 = 70 - 20e^{-kt}$$

$$20e^{-kt} = 10$$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln(\frac{1}{2})$$

$$t = \frac{\ln 2}{k} = 31.08 \text{ mins}$$