Autonomous Differential Equations

A first-order differential equation with independent variable t and dependent variable y is **autonomous** if

$$\frac{dy}{dt} = f(y).$$

The rate of change of $y(t)$ depends only on the value of y.

Examples of autonomous equations: exponential growth model, radioactive decay, logistic population model

Example. $\dfrac{dv}{dt} = -kv + a\sin bt$

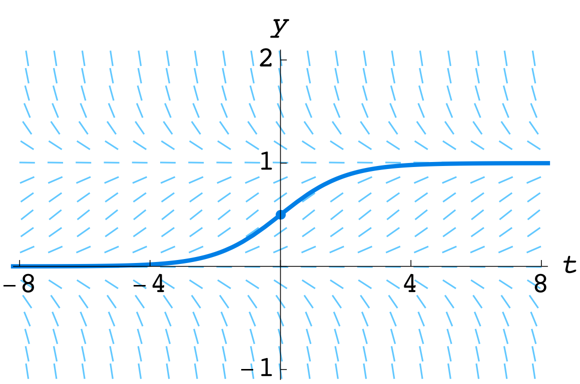
This is a nonautonomous linear differential equation that is related to simple models of voltage in an electric circuit (k, a, and b are parameters).

Comments:

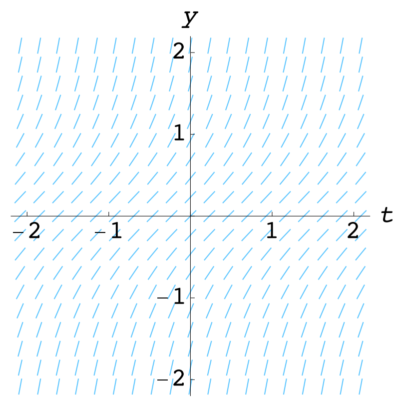
1. Many interesting models in science and engineering are autonomous (but not every model).
2. Every autonomous equation is separable, but the integrals may be impossible to calculate in terms of standard functions.

Basic Fact: Given the graph of one solution to an autonomous equation, we can get the graphs of many other solutions by translating that graph left or right.

Example 1. $\dfrac{dy}{dt} = y(1-y)$

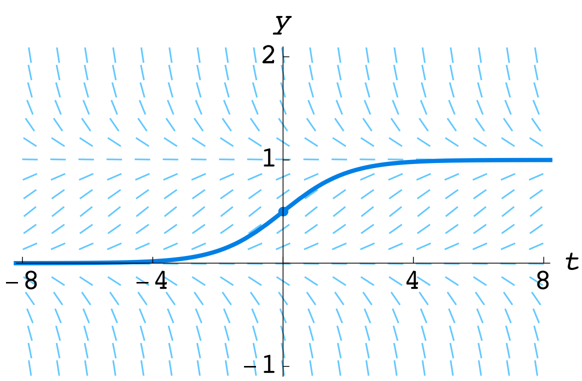


Example 2. $\dfrac{dy}{dt} = 1 + y^2$



The slope field has so much redundant information that we can replace it with the **phase line**. Here's the phase line for our standard example:

Example. $\dfrac{dy}{dt} = y(1 - y)$

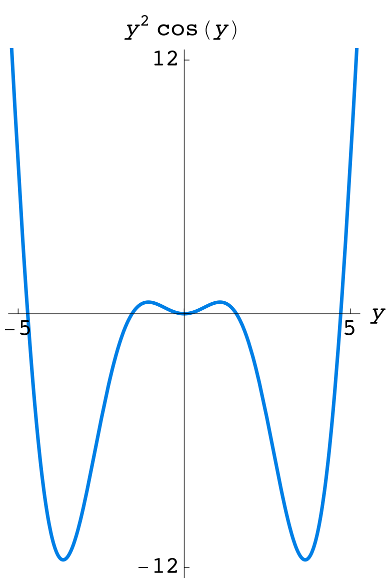


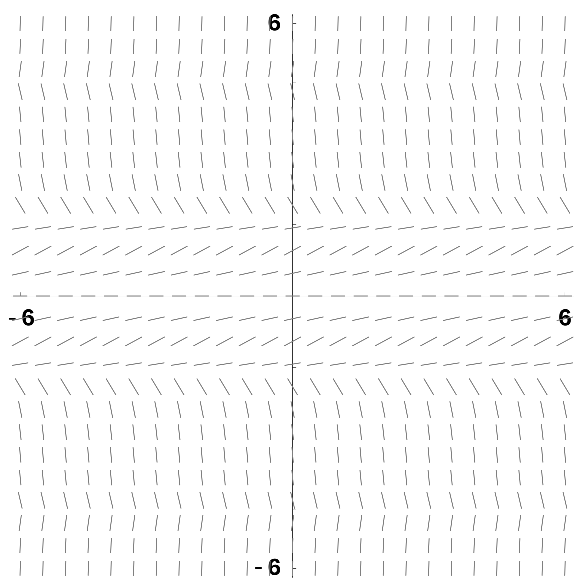
I've built an animation that illustrates how you should interpret this phase line. Also, `PhaseLines` in `DETools` helps you visualize the meaning of the phase line.

Building phase lines

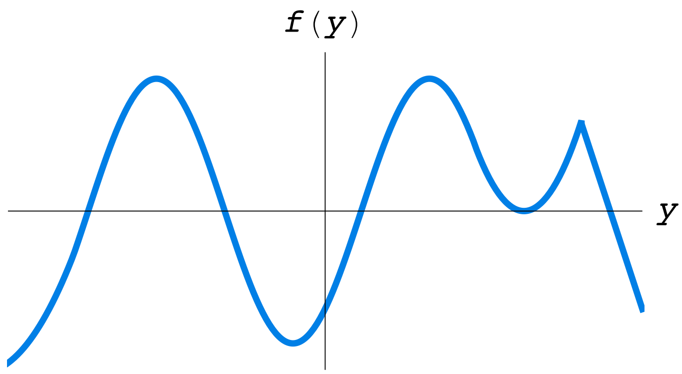
How do we go about building a phase line from a differential equation?

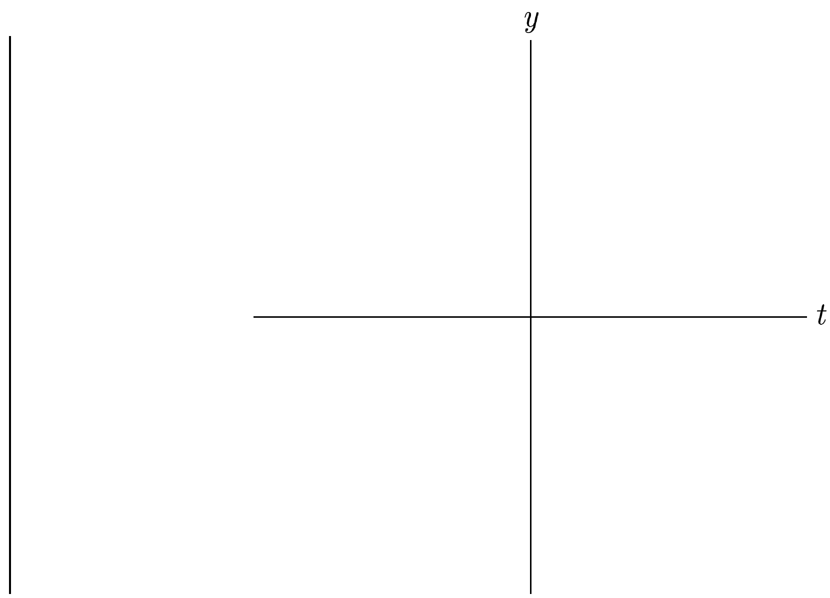
Example 1. $\dfrac{dy}{dt} = y^2 \cos y$





Example 2. $\dfrac{dy}{dt} = f(y)$ where $f(y)$ is given by the graph





Linear differential equations

A first-order differential equation

$$\frac{dy}{dt} = f(t, y)$$

with independent variable t and dependent variable y is **linear** if it can be written as

$$\frac{dy}{dt} = a(t)y + b(t).$$

In other words, the *dependent* variable only appears linearly in the equation.

Linear differential equations:

$$\frac{dy}{dt} = 5y$$

$$\frac{dy}{dt} = (\cos t)y$$

$$\frac{dy}{dt} = y - t^2$$

Nonlinear differential equations:

$$\frac{dy}{dt} = t \cos y$$

$$\frac{dy}{dt} = y^2 - t$$

The linear differential equation

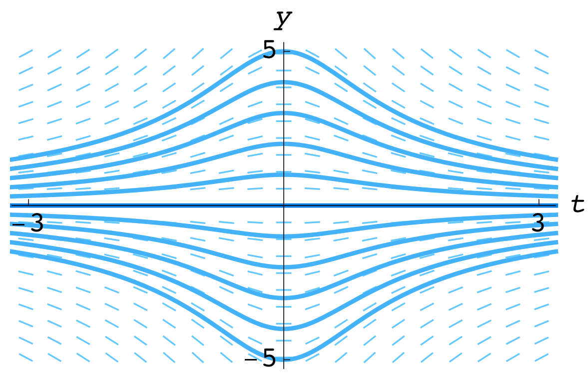
$$\frac{dy}{dt} = a(t)y + b(t)$$

is **homogeneous** if $b(t) = 0$ for all t. Otherwise, it is **nonhomogeneous**. (Some people use the term inhomogeneous.)

Where have we seen homogeneous linear differential equations before?

Example. $\dfrac{dy}{dt} = \dfrac{-ty}{1+t^2}$

(Graphs and slope field on top of next page.)



Linearity Principles

Why are linear equations so much more amenable to analytic techniques than nonlinear equations? The reason is that their solutions satisfy important linearity principles.

Let's begin with homogeneous linear equations:

Linearity Principle. If $y_h(t)$ is a solution of a homogeneous linear differential equation

$$\frac{dy}{dt} = a(t)\, y,$$

then any *constant* multiple $y_k(t) = k\, y_h(t)$ of $y_h(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a nonzero solution $y_h(t)$, we obtain another solution by multiplying $y_h(t)$ by k.

Note that the Linearity Principle is not true for nonlinear equations. For example, consider

$$\frac{dy}{dt} = y^2.$$

Check that one solution is

$$y_1(t) = \frac{1}{1-t},$$

and then check that

$$y_2(t) = 2y_1(t) = \frac{2}{1-t}$$

is not a solution.

There is a similar "linearity" principle for nonhomogeneous linear equations:

Extended Linearity Principle For First-Order Equations. Consider a first-order, nonhomogeneous, linear equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ ("p" for particular solution) is *any* solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

We can paraphrase the Extended Linearity Principle by saying that:

> The general solution of a nonhomogeneous linear equation consists of the sum of *any* particular solution of the nonhomogeneous equation and the general solution of the associated homogeneous equation.