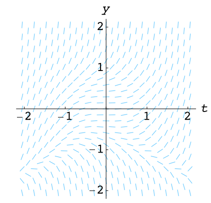
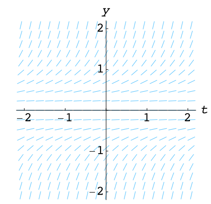
Existence and Uniqueness Theory

First we consider three examples to illustrate the idea of the domain of a differential equation:

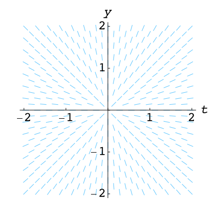
Example 1. $\dfrac{dy}{dt} = y^3 + t^2$



Example 2. $\dfrac{dy}{dt} = y^2$



Example 3. $\dfrac{dy}{dt} = \dfrac{y}{t}$



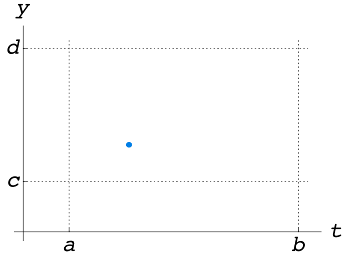
We start our discussion of the theory with the Existence Theorem:

Existence Theorem Suppose $f(t,y)$ is a continuous function in a rectangle of the form

$$\{(t,y) \mid a < t < b,\ c < y < d\}$$

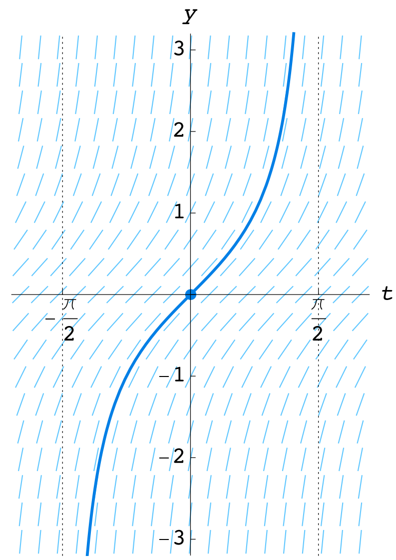
in the ty-plane. If (t_0, y_0) is a point in this rectangle, then there exists an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ that solves the initial-value problem

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0. \quad \blacksquare$$



What's the significance of the ϵ in the Existence Theorem?

Example. $\dfrac{dy}{dt} = 1 + y^2, \quad y(0) = 0$



What does the Existence Theorem tell us about the initial-value problem

$$\frac{dy}{dt} = y^3 + t^2, \quad y(0) = 0?$$

The other main theoretical result in differential equations is the Uniqueness Theorem.

Uniqueness Theorem Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle of the form

$$\{(t, y) \mid a < t < b,\ c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

for all t in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ (where ϵ is some positive number), then

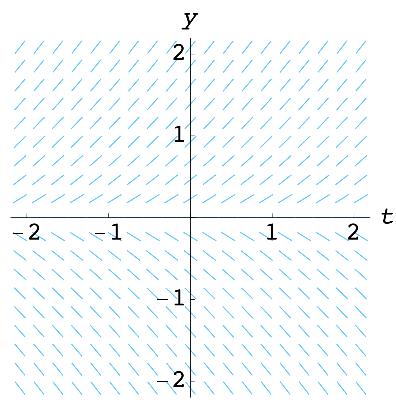
$$y_1(t) = y_2(t)$$

for $t_0 - \epsilon < t < t_0 + \epsilon$. That is, the solution to the initial-value problem is *unique*. ∎

Here's an example that lacks uniqueness:

Example. $\dfrac{dy}{dt} = \sqrt[3]{y}$

(More blank space and the slope field for $dy/dt = \sqrt[3]{y}$ on the top of the next page.)



Bogus Example. The example

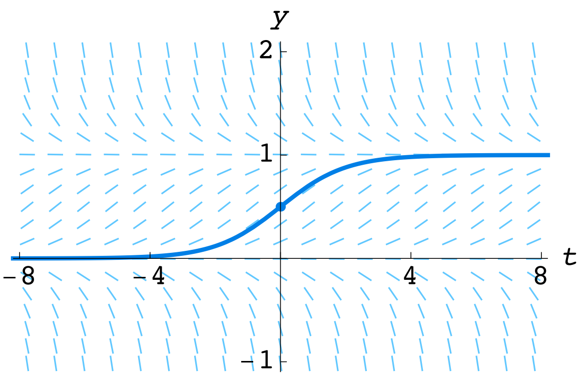
$$\frac{dy}{dt} = \frac{y}{t} + t\cos t$$

in `FirstOrderSystems` seems to violate the Uniqueness Theorem, but in fact it does not. Why?

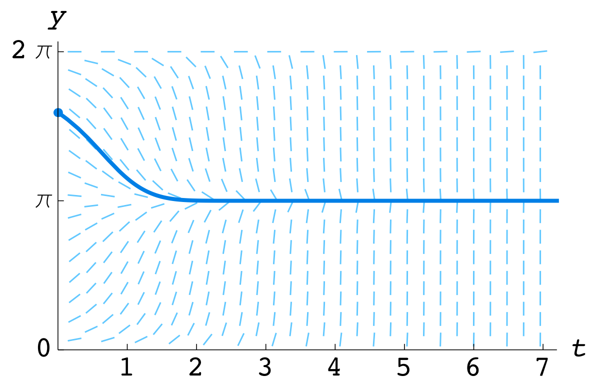
The Uniqueness Theorem has many useful consequences. Here are three examples:

Example 1. $\dfrac{dy}{dt} = -2ty^2$

Example 2. $\dfrac{dy}{dt} = y(1-y)$



Example 3. $\dfrac{dy}{dt} = e^t \sin y$



Autonomous Differential Equations

A first-order differential equation with independent variable t and dependent variable y is **autonomous** if

$$\frac{dy}{dt} = f(y).$$

The rate of change of $y(t)$ depends only on the value of y.

Examples of autonomous equations: exponential growth model, radioactive decay, logistic population model

Example. $\frac{dv}{dt} = -kv + a \sin bt$

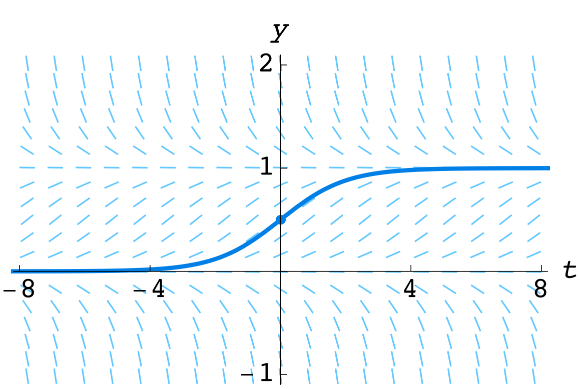
This is a nonautonomous linear differential equation that is related to simple models of voltage in an electric circuit (k, a, and b are parameters).

Comments:

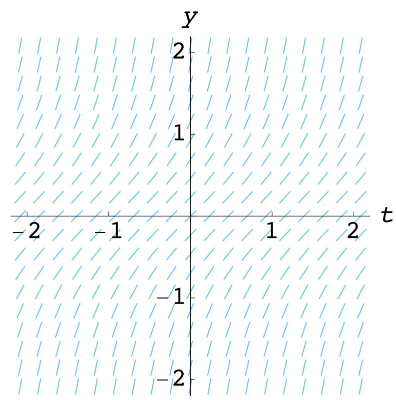
1. Many interesting models in science and engineering are autonomous (but not every model).
2. Every autonomous equation is separable, but the integrals may be impossible to calculate in terms of standard functions.

Basic Fact: Given the graph of one solution to an autonomous equation, we can get the graphs of many other solutions by translating that graph left or right.

Example 1. $\frac{dy}{dt} = y(1 - y)$

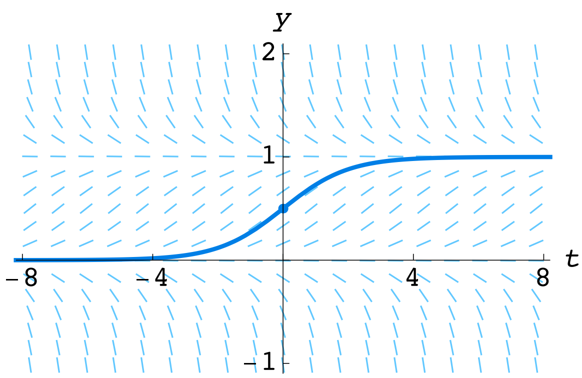


Example 2. $\dfrac{dy}{dt} = 1 + y^2$



The slope field has so much redundant information that we can replace it with the **phase line**. Here's the phase line for our standard example:

Example. $\dfrac{dy}{dt} = y(1 - y)$



I've built an animation that illustrates how you should interpret this phase line. Also, `PhaseLines` in `DETools` helps you visualize the meaning of the phase line.