A little review and fixing an omission

Consider the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with independent variable t and dependent variables x and y. We use the right-hand side of this system to form a vector field

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

in the xy-plane. We also use $x(t)$ and $y(t)$ to form a vector-valued function

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then the (scalar) system of differential equations can be rewritten as one vector differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

Example 1 revisited. Let's consider the simple mass-spring system with $k = m$, but this time we'll write it using the variables x and y to be consistent with the `HPGSystemSolver` notation. We have

$$\frac{d^2x}{dt^2} + x = 0.$$

The equivalent first-order system is

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x.$$

and, the vector field is

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

We guessed some solutions to the second-order equation before spring break. One is $x(t) = \cos t$. The corresponding vector function is

$$\mathbf{Y}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

Then

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} \quad \text{and} \quad \mathbf{F}\left(\mathbf{Y}(t)\right) = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}.$$

Damped Harmonic Oscillator

Let's return to our mass-spring system and add a term that models damping.

Assumption: The damping force is proportional to the speed of the mass and it acts as a restoring force.

This second-order equation and its equivalent system appear in many applications. In `DETools`, you will find it in `MassSpring` and `RLCCircuits`, and it has also been used to study biological processes such as the blood glucose regulatory system in humans.

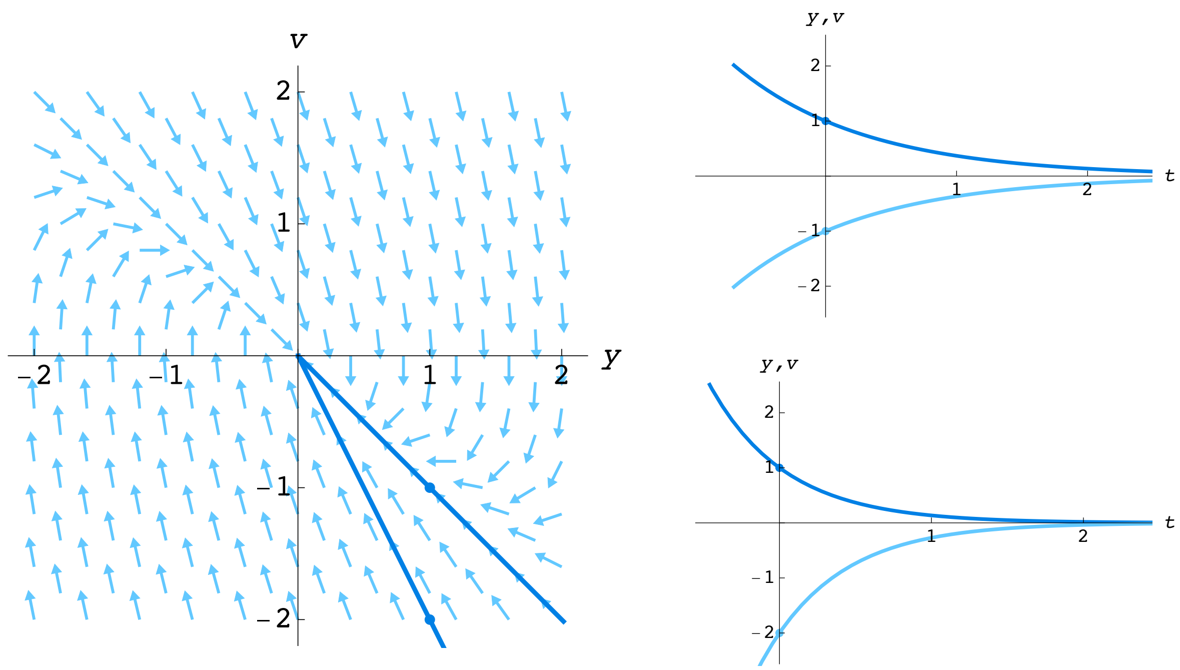
There is a guessing technique for the damped harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

Example. Consider the harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0.$$

Its characteristic equation is



Analytic Techniques:

There are few analytic techniques that work for both linear and nonlinear systems.

1. You can always check to see if a given function is a solution (no wrong answers).
2. General solution of a partially-decoupled system

   Example. Consider the system

   $$\frac{dx}{dt} = 2y - x$$
   $$\frac{dy}{dt} = y.$$

   We can calculate the general solution using methods we learned for first-order equations:

Euler's method for a system

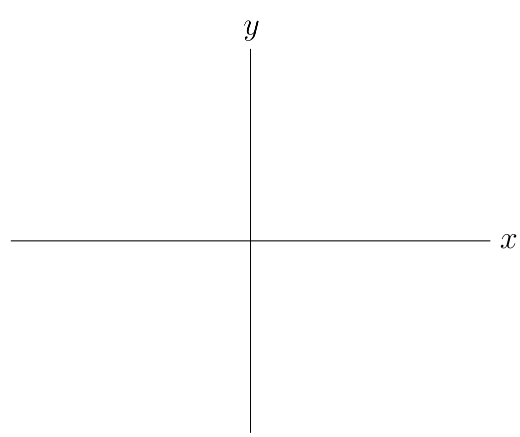
We can use the vector field for a system to produce numerical approximations for the solutions.

Example. Consider the initial-value problem

$$\frac{dx}{dt} = -y \qquad (x_0, y_0) = (2, 0).$$
$$\frac{dy}{dt} = x - y$$

The `EulersMethodForSystems` tool demonstrates the method. We pick a large step size $\Delta t = 0.5$ so that we can see the method in action.

k	x_k	y_k	m_k	n_k
0	2	0		
1				
2				
3				



Here's the general formula for Euler's method written in vector notation:

Let $\mathbf{Y}_0$ be an initial condition and Δt be a step size. Consider the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}), \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

Then $t_{k+1} = t_k + \Delta t$ and $\mathbf{Y}_{k+1} = \mathbf{Y}_k + (\Delta t)\,\mathbf{F}(\mathbf{Y}_k)$. There are spreadsheets on the course web site that implement the method.

Existence and Uniqueness Theory for Systems

There is an existence and uniqueness theorem for systems just like the theorem for equations.

Existence and Uniqueness Theorem. Let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y})$$

be a system of differential equations. Suppose that t_0 is an initial time and $\mathbf{Y}_0$ is an initial value. Suppose also that the function $\mathbf{F}$ is continuously differentiable. Then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}(t)) \quad \text{and} \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

In other words, $\mathbf{Y}(t)$ satisfies the initial-value problem. Moreover, for t in this interval, this solution is unique.

There is an important consequence of the Uniqueness Theorem for autonomous systems: Consider the metaphor of the parking lot.