Existence and Uniqueness Theory for Systems

There is an existence and uniqueness theorem for systems just like the theorem for equations.

Existence and Uniqueness Theorem. Let

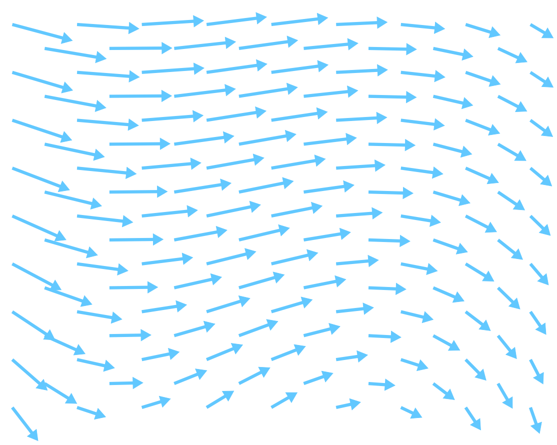
$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y})$$

be a system of differential equations. Suppose that t_0 is an initial time and $\mathbf{Y}_0$ is an initial value. Suppose also that the function $\mathbf{F}$ is continuously differentiable. Then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}(t)) \quad \text{and} \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

In other words, $\mathbf{Y}(t)$ satisfies the initial-value problem. Moreover, for t in this interval, this solution is unique.

There is an important consequence of the Uniqueness Theorem for autonomous systems: Consider the metaphor of the parking lot.



Given the autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

Let $\mathbf{Y}_0$ be an initial condition such that $\mathbf{Y}_1(t)$ is a solution that satisfies $\mathbf{Y}(t_1) = \mathbf{Y}_0$ and $\mathbf{Y}_2(t)$ is another solution that satisfies $\mathbf{Y}(t_2) = \mathbf{Y}_0$. Then

$$\mathbf{Y}_2(t) = \mathbf{Y}_1(t - (t_2 - t_1)).$$

Example. Consider the second-order equation

$$\frac{d^2x}{dt^2} + x = 0,$$

which is equivalent to the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x.$$

Note that

$$\mathbf{Y}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

are both solutions to the system. How are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ related?

Here is an informal restatement of this consequence of uniqueness:

> For an autonomous system, if two solution curves in the phase plane touch, then they are identical.

Linear systems

Linear systems and second-order linear equations are the most important systems we study in this course.

What is a linear system with two dependent variables?

What is a second-order, homogeneous, linear equation?

Linear systems written in vector notation suggest the use of matrix multiplication:

Recall two examples that we have already discussed.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\frac{dx}{dt} = 2y - x$$
$$\frac{dy}{dt} = y.$$

It is

$$x(t) = y_0 e^t + (x_0 - y_0)e^{-t}$$
$$y(t) = y_0 e^t.$$

or in vector form

$$\mathbf{Y}(t) = e^t \begin{pmatrix} y_0 \\ y_0 \end{pmatrix} + e^{-t} \begin{pmatrix} x_0 - y_0 \\ 0 \end{pmatrix}.$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

and its equivalent system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -2y - 3v,$$

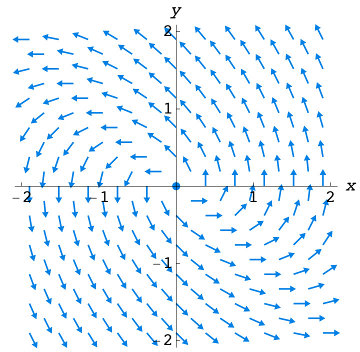
we used a guessing technique to find two (scalar) solutions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$. In vector form, these solutions are written as

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

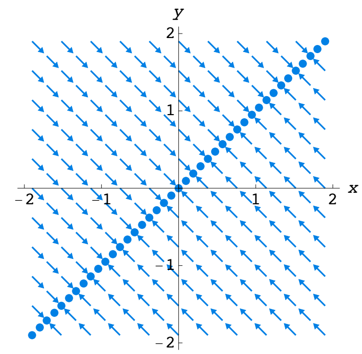
Given a linear system $d\mathbf{Y}/dt = \mathbf{AY}$, how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of the linear system $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$?

Example. Let $\mathbf{A}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.



Example. Let $\mathbf{A}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.



Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.

The Linearity Principle

Let's return to Example 1. For practice, we'll use vector notation this time:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

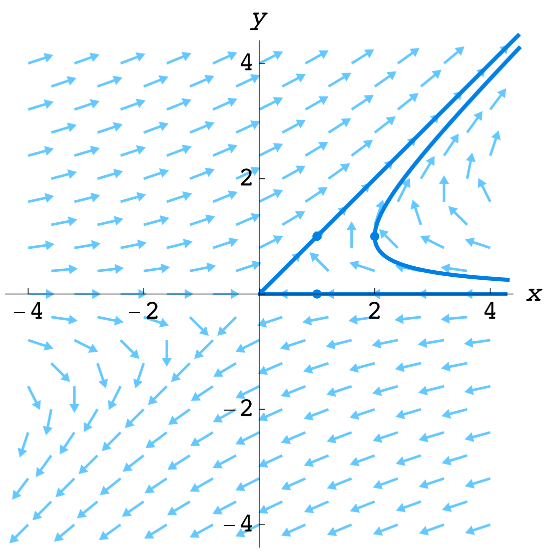
Also consider three different initial conditions

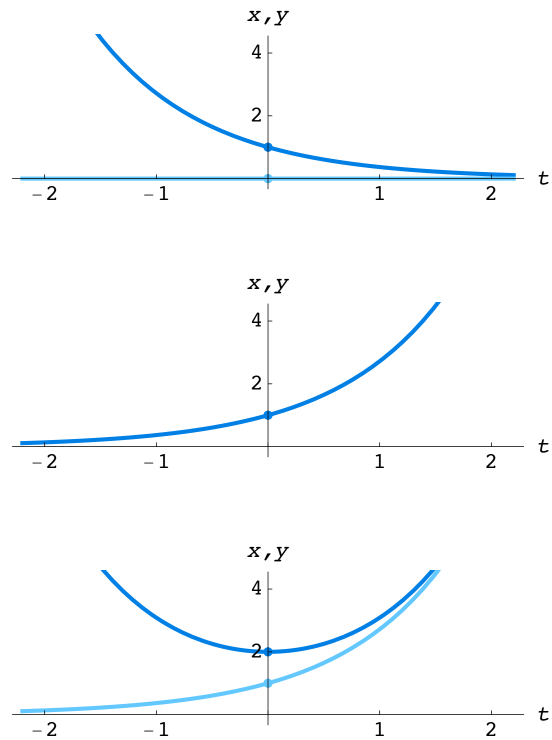
$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

They correspond to the three solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_3(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}.$$

Let's see what happens when we graph these solutions.





How are these three solutions related?

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

This principle gives us a more general way to find solutions of linear systems. To see how this approach works, let's consider Example 1 again along with the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.