One more comment on equilibrium points

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.

The Linearity Principle

Let's return to Example 1 from last class. For practice, we'll use vector notation this time:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

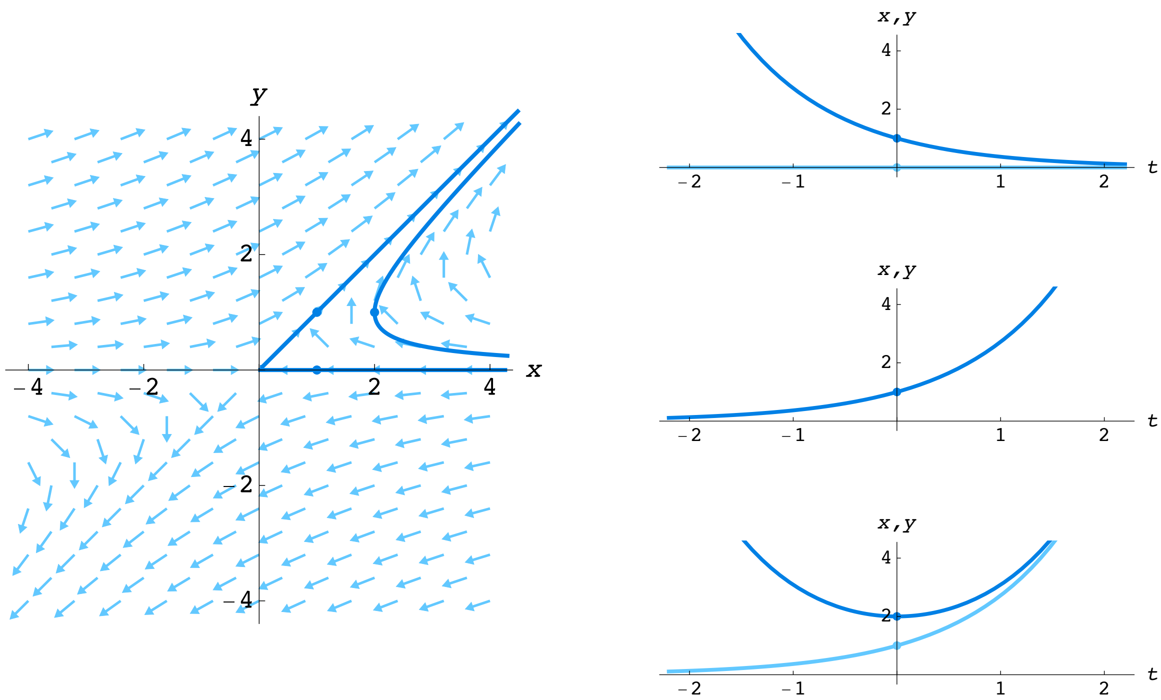
Also consider three different initial conditions

$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

They correspond to the three solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_3(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}.$$

Let's see what happens when we graph these solutions.



How are these three solutions related?

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

This principle gives us a more general way to find solutions of linear systems. To see how this approach works, let's consider Example 1 again along with the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

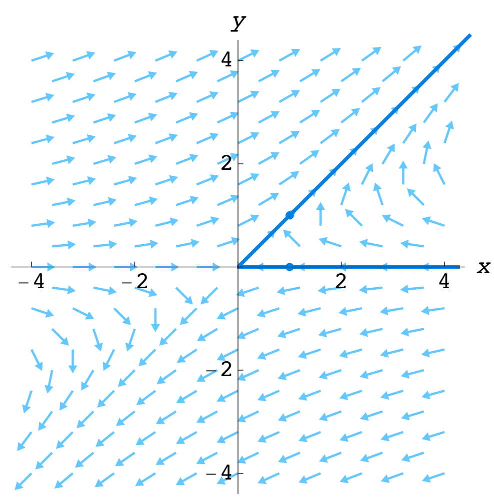
Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

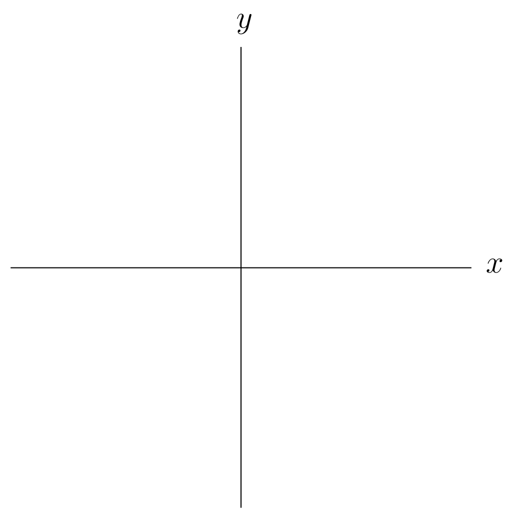
For an arbitrary linear system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, how many solutions do we need to solve every initial-value problem?

Questions: How do we find two linearly independent solutions? Is there something special about the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ in the example?



Straight-line solutions

For a general linear system of the form $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, what geometric property of the vector field guarantees the existence of these "straight-line" solutions?



"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function $\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$ is a solution to the linear differential equation $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.

We want nonzero initial conditions $\mathbf{Y}_0$ (vectors) so that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

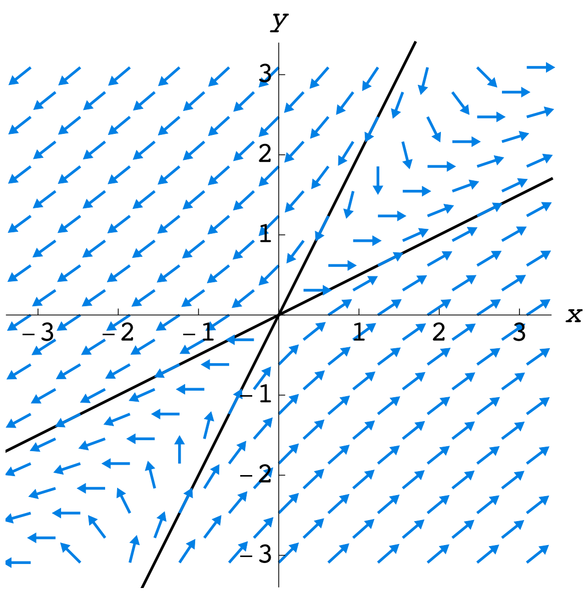
for some scalar λ.

Terminology: The scalar λ is called an *eigenvalue* of the matrix $\mathbf{A}$ and the vector $\mathbf{Y}_0$ is called an *eigenvector* associated to the eigenvalue λ.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.



Aside from the theory of algebraic linear equations

For what matrices $\mathbf{B}$ does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices. The matrix equation $\mathbf{BY} = \mathbf{0}$ has nontrivial solutions $\mathbf{Y}$ if and only if $\det \mathbf{B} = 0$.

Notes:

1. Most matrices are nonsingular (not singular).
2. We encountered a singular matrix last class when we studied the linear system that had a line of equilibrium points.

Finding eigenvalues and eigenvectors:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{Y}.$$