Straight-line solutions and eigenstuff

Last class we learned that we need two linearly independent solutions of a 2D-linear system to obtain the general solution. Moreover, we learned that initial conditions $\mathbf{Y}_0$ that satisfy the equation

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

yield especially nice solutions. These solutions

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

are called *straight-line solutions*.

We want nonzero initial conditions $\mathbf{Y}_0$ (vectors) so that

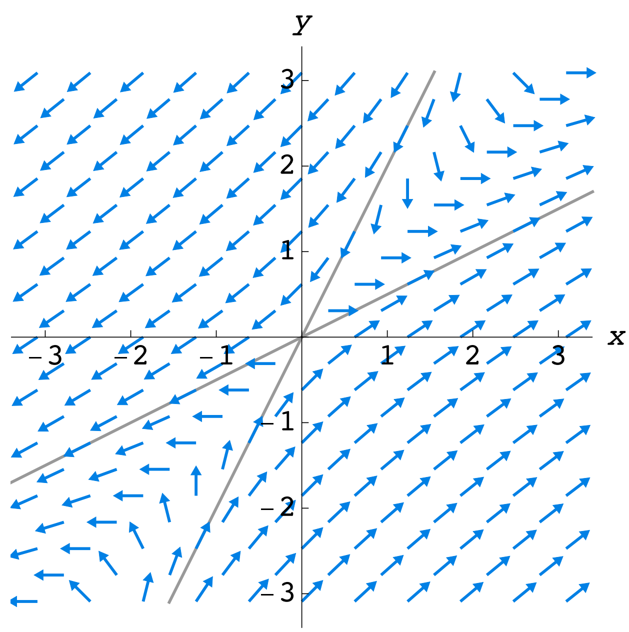
$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some scalar λ.

Terminology: The scalar λ is called an *eigenvalue* of the matrix $\mathbf{A}$ and the vector $\mathbf{Y}_0$ is called an *eigenvector* associated to the eigenvalue λ.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$



Note that

$$\mathbf{A}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for any scalar λ.

How do we find these special directions?

Aside from the theory of algebraic linear equations

For what matrices $\mathbf{B}$ does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices. The matrix equation $\mathbf{BY} = \mathbf{0}$ has nontrivial solutions $\mathbf{Y}$ if and only if $\det \mathbf{B} = 0$.

Notes:

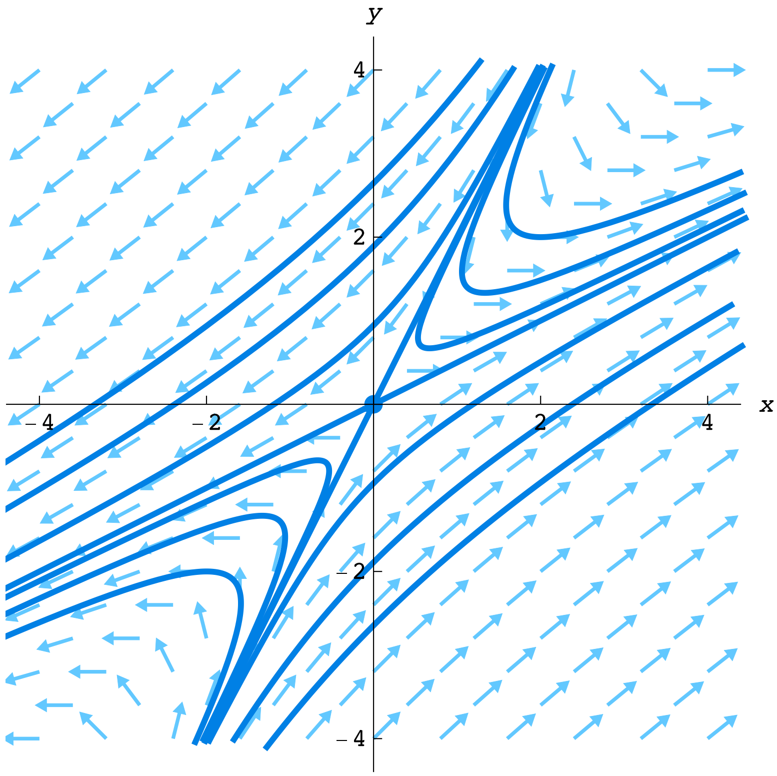
1. Most matrices are nonsingular (not singular).
2. We encountered a singular matrix when we studied the linear system that had a line of equilibrium points.

Finding eigenvalues and eigenvectors:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{Y}.$$

Here's the phase portrait for this system:



Facts about eigenvalues and eigenvectors: Given a 2×2 matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.
2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to λ.
3. Eigenvectors associated to distinct eigenvalues are linearly independent.

Summary of Case of Two Distinct Real Eigenvalues

Suppose $\mathbf{A}$ is a matrix with two eigenvalues λ_1 and λ_2. To be consistent, we will assume that $\lambda_1 < \lambda_2$, that $\mathbf{V}_1$ is an eigenvector associated to λ_1, and that $\mathbf{V}_2$ is an eigenvector associated to λ_2. The general solution of

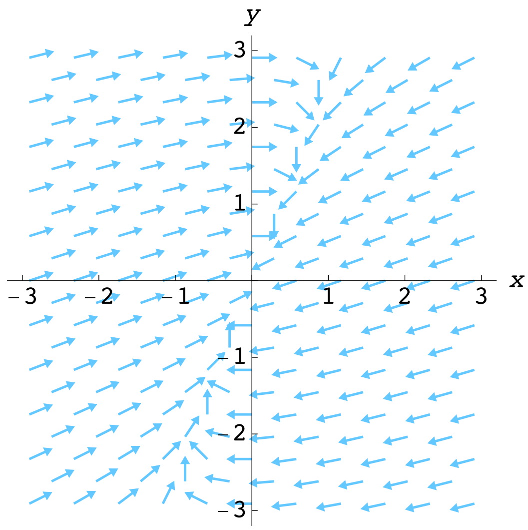
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is $\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$.

Case 1: $\lambda_1 < \lambda_2 < 0$.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$



Sketching component graphs

Once we understand the phase portrait, we should also be able to sketch the component graphs without `HPGSystemSolver`.

For example, once again consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

Let's sketch the $x(t)$- and $y(t)$-graphs that correspond to the initial conditions $(-3, 2)$ and $(3, 2)$.