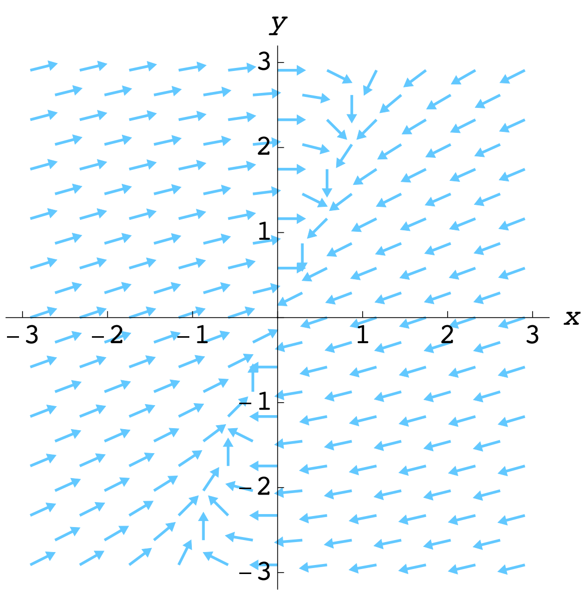
More on the example from last class

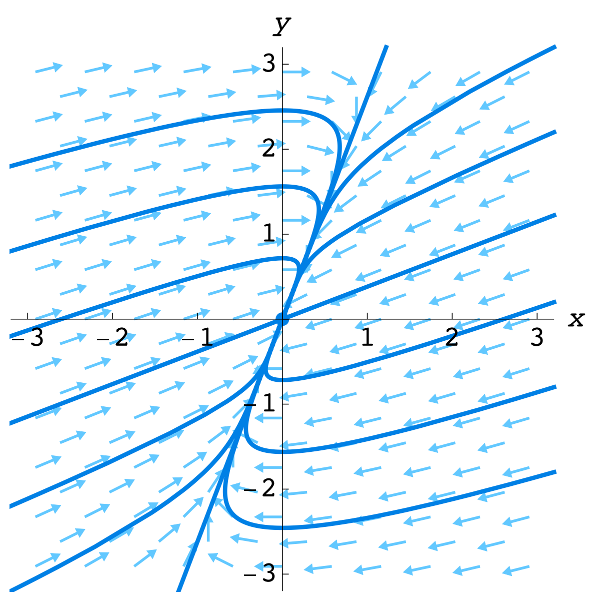
Example. Once again consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

For this example, the eigenvalues are $\lambda = \frac{1}{2}(-3 \pm \sqrt{5}\,)$. Both are negative.

The slope of the eigenline that corresponds to the "fast" eigenvalue $\lambda_1 = \frac{1}{2}(-3 - \sqrt{5}\,)$ is approximately 0.4, and the slope of the eigenline that corresponds to the "slow" eigenvalue $\lambda_2 = \frac{1}{2}(-3 + \sqrt{5}\,)$ is approximately 2.6.



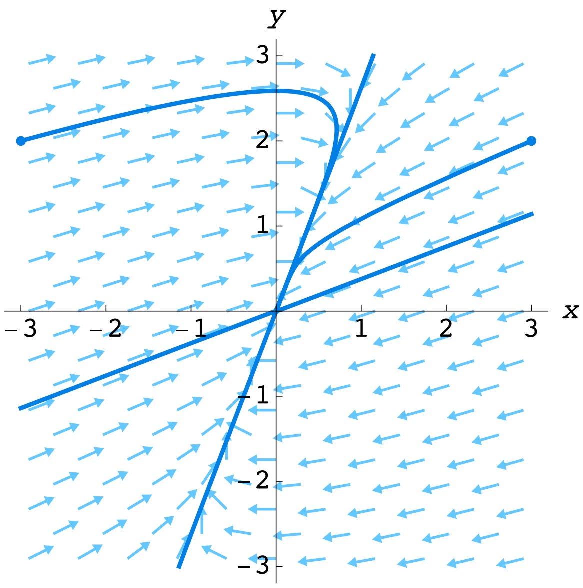


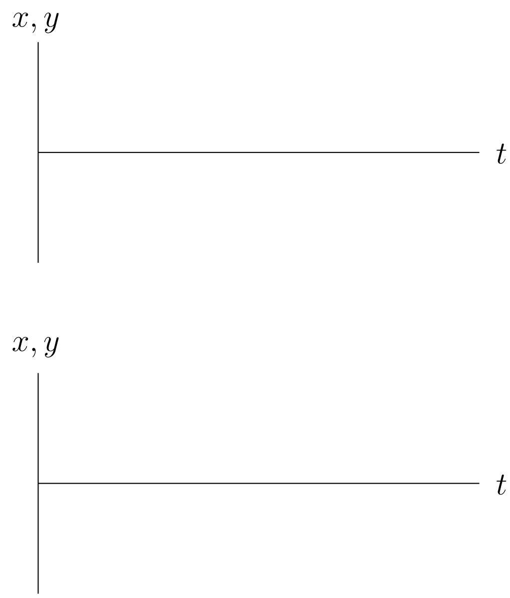
Sketching component graphs

Once we understand the phase portrait, we should also be able to sketch the component graphs without `HPGSystemSolver`.

Let's sketch the $x(t)$- and $y(t)$-graphs that correspond to the initial conditions $(-3, 2)$ and $(3, 2)$ for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$



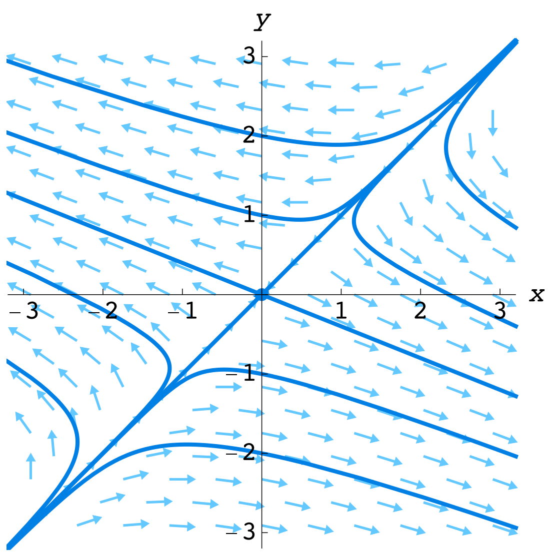


Case 2: $\lambda_1 < 0 < \lambda_2$.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 6$. The λ_1-eigenline is the diagonal line $y_1 = x_1$, and the λ_2-eigenline is the line $y_2 = -\frac{2}{5}x_2$.



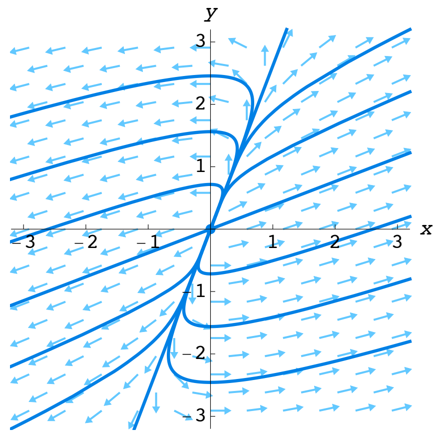
Case 3: $0 < \lambda_1 < \lambda_2$.

Example. Consider $d\mathbf{Y}/dt = \mathbf{B}\mathbf{Y}$ where

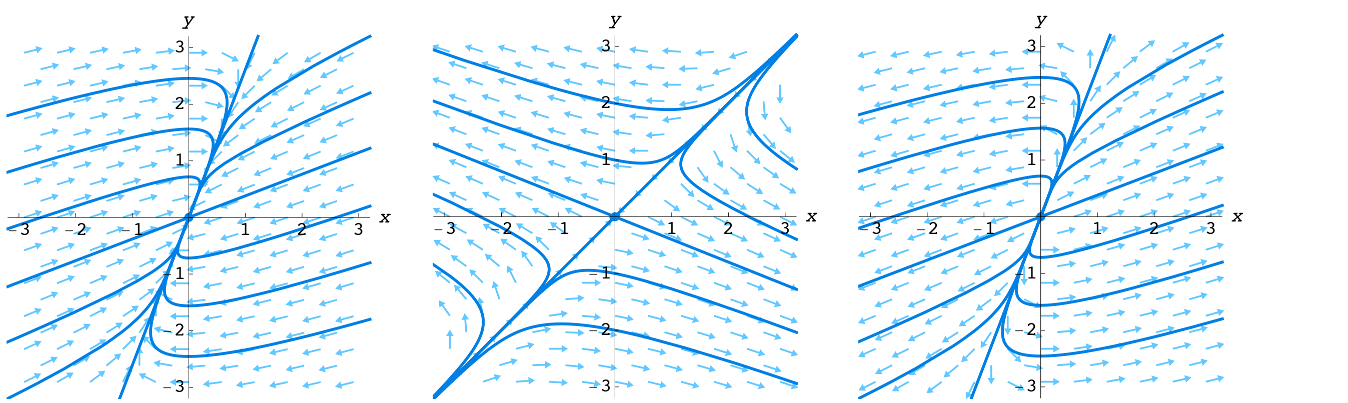
$$\mathbf{B} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\mathbf{B} = -\mathbf{A}$ where $\mathbf{A}$ is the matrix used in the example on page 1. The eigenvalues of $\mathbf{B}$ are $\lambda = \frac{1}{2}(3 \pm \sqrt{5}\,)$. Both are positive.

The slope of the eigenline that corresponds to the "fast" eigenvalue $\lambda_1 = \frac{1}{2}(3 + \sqrt{5}\,)$ is approximately 0.4, and the slope of the eigenline that corresponds to the "slow" eigenvalue $\lambda_2 = \frac{1}{2}(3 - \sqrt{5}\,)$ is approximately 2.6.



Summary for real and distinct (nonzero) eigenvalues



sink ($\lambda_1 < \lambda_2 < 0$) saddle ($\lambda_1 < 0 < \lambda_2$) source ($0 < \lambda_1 < \lambda_2$)

Complex eigenvalues

What happens if the eigenvalues of a linear system are complex numbers?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}.$$

Let's see that happens if we take a look at this system using `MatrixFields`, and then we'll compute the eigenstuff for this matrix.

Eigenvalues:

Eigenvectors:

(Lots of blank space on the next page.)

We now have a complex-valued solution of the form

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

There are lots of questions that come with this formula. First, what does the formula mean? Second, what good is it given that we are interested in real-valued solutions to our linear systems?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Let's use this series where $x = bi$.

We use Euler's formula

$$e^{bi} = \cos b + i \sin b$$

applied to the complex-valued function $e^{(a+bi)t}$.

But why does this help us solve our differential equation?

Theorem. Consider $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.

Now we can derive the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}$$

using the complex-valued solution $\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$.