Last class we studied an example with complex eigenvalues.

Example. Consider $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$.

Its eigenvalues are $\lambda = -2 \pm i$, and by solving the eigenvector equation $\mathbf{AY}_0 = (-2+i)\mathbf{Y}_0$, we derived the "straight-line solution"

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

There are lots of questions that come with this formula. First, what does the formula mean? Second, what good is it given that we are interested in real-valued solutions to our linear systems?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Let's use this series where $x = bi$.

We use Euler's formula

$$e^{bi} = \cos b + i \sin b$$

applied to the complex-valued function $e^{(a+bi)t}$.

But why does this help us solve our differential equation?

Theorem. Consider $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.

Now we can derive the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}$$

using the complex-valued solution $\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$.

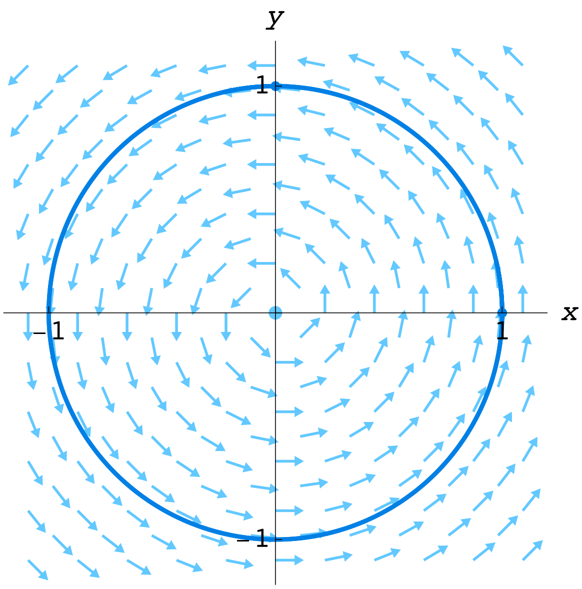
The geometry of complex eigenvalues

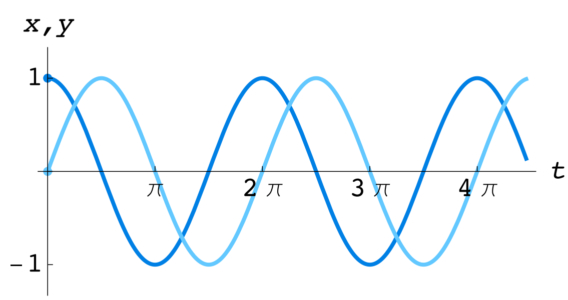
Example 1. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$ where $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

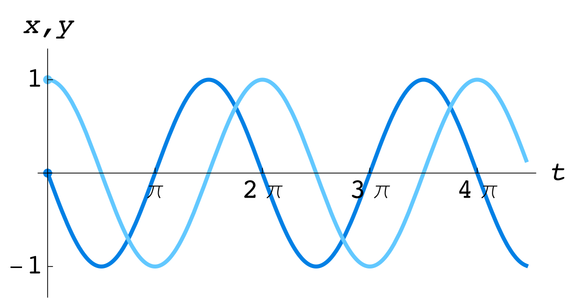
The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 1$, so the eigenvalues are $\lambda = \pm i$. One eigenvector associated to the eigenvalue $\lambda = i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

A solution curve and two pairs of $x(t)$- and $y(t)$-graphs are shown below.





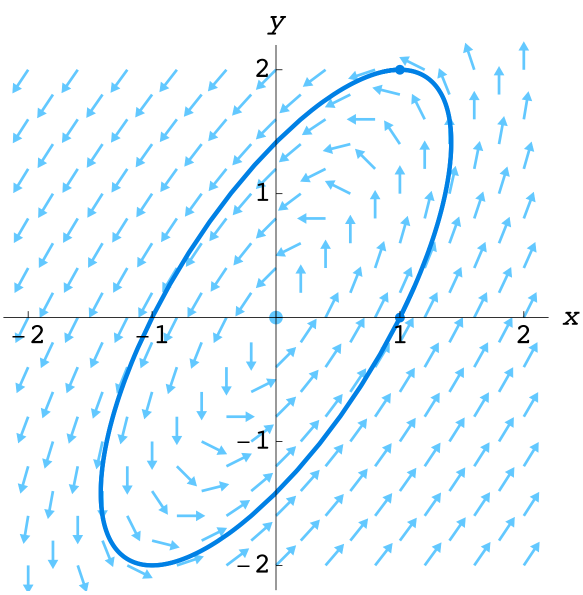


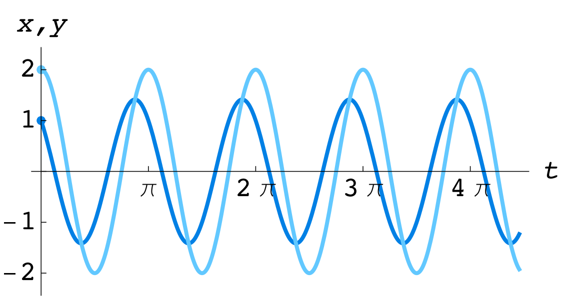
Example 2. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y}$ where $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}$.

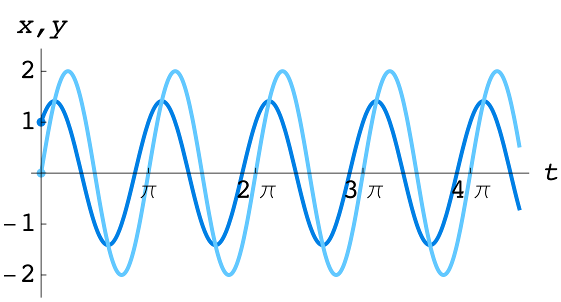
The characteristic polynomial of $\mathbf{B}$ is $\lambda^2 + 4$, so the eigenvalues are $\lambda = \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = 2i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

We get ellipses centered at the origin in the phase plane.



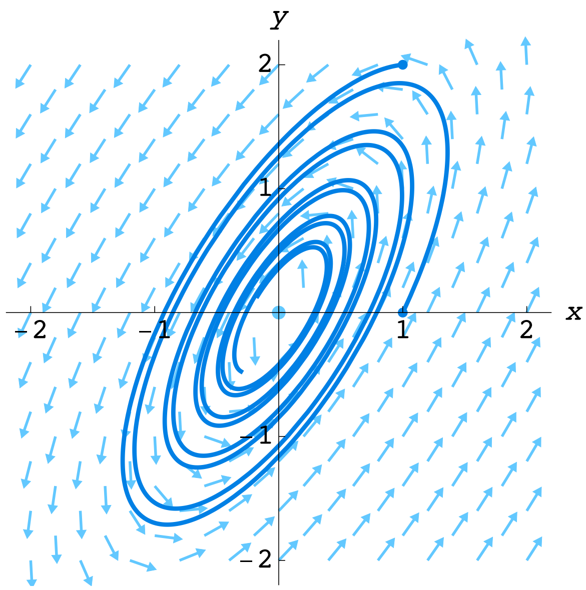


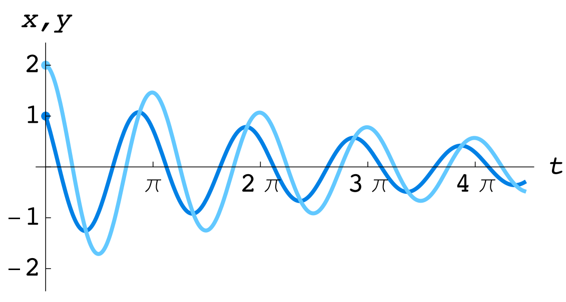


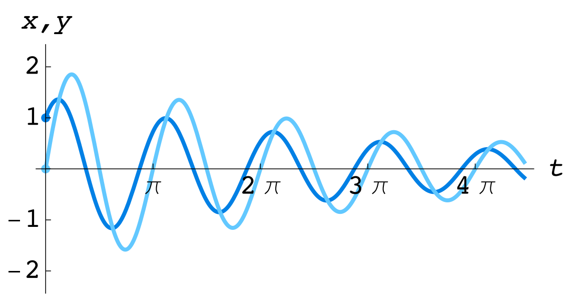
Example 3. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y}$ where $\mathbf{C} = \begin{pmatrix} 1.9 & -2 \\ 4 & -2.1 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{C}$ is $\lambda^2 + 0.2\lambda + 4.01$, so the eigenvalues are $\lambda = -0.1 \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = -0.1 + 2i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

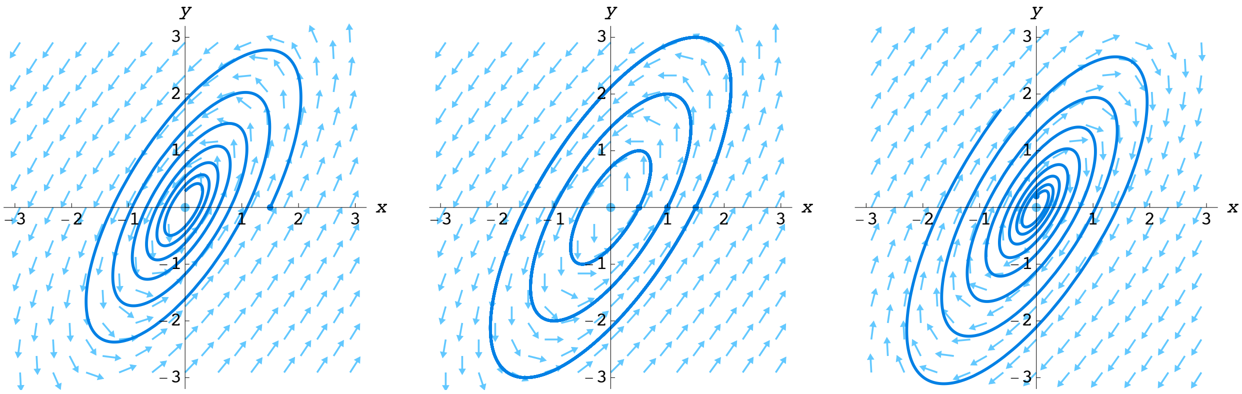






Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

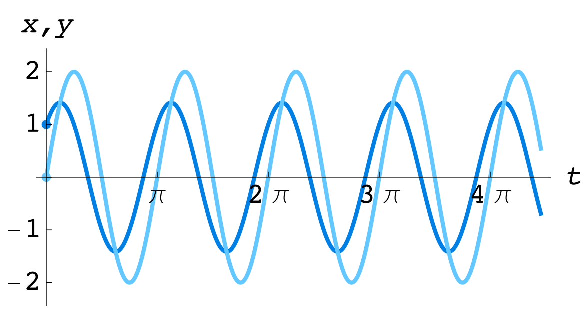
Here are the possible phase portraits:



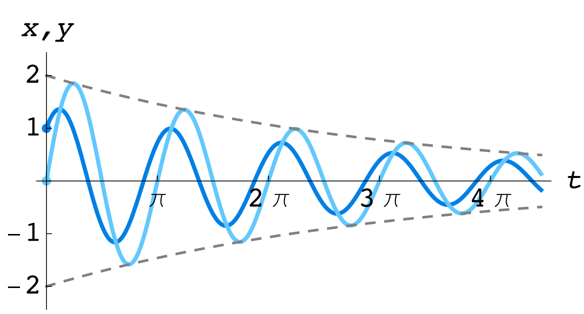
spiral sink ($a < 0$) center ($a = 0$) spiral source ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time T such that

$$x(t+T) = x(t) \quad \text{and} \quad y(t+T) = y(t)$$

for all t. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

Definition. The *frequency* F of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" T. What is its frequency F?

Example. Consider the standard sinusoidal functions $g(t) = \cos \beta t$ and $g(t) = \sin \beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let's denote the angular frequency by f. Then

$$f = 2\pi F.$$