Predator-prey versus mass-spring

Example. The predator-prey system, a 2D first-order autonomous system,

$$\frac{dR}{dt} = aR - bRF$$
$$\frac{dF}{dt} = -cF + dRF.$$

Example. The mass-spring system, a second-order autonomous equation,

$$m\frac{d^2y}{dt^2} + ky = 0.$$

Although they seem quite different, they have more in common than one might think. In particular, the mass-spring system can also be written as a first-order system by introducing the “new” variable v (which is just dy/dt). We get

In what ways is the mass-spring system similar to the predator-prey system?

An initial condition for the predator-prey system is a pair (R_0, F_0) of population values.

An initial condition for the mass-spring system is also a pair (y_0, v_0). The first number indicates the initial displacement and the second number indicates the initial velocity.

Since the mass-spring system can be expressed as a system, all of the terms that we discussed for the predator-prey system apply to the mass-spring system as well (equilibrium solutions, component graphs, phase plane, solution curve, phase portrait, . . .).

One way that the two systems differ is by the fact that we can find formulas for the solutions of the mass-spring system but not for the predator-prey system.

For example, consider the special case of the mass-spring system where $k = m$. We get

$$\frac{d^2y}{dt^2} = -y,$$

and we can guess some solutions to this equation:

The equivalent system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -y \end{aligned}$$

has one $(y(t), v(t))$ pair of solutions for each solution to the second-order equation:

What are the initial conditions for these solutions?

The vector field of an autonomous system

We get a better geometric understanding of the solutions of a first-order system of differential equations if we rewrite the system as a vector equation that applies to a vector-valued function.

Consider the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with independent variable t and dependent variables x and y. We use the right-hand side of this system to form a vector field

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

in the xy-plane. We also use $x(t)$ and $y(t)$ to form a vector-valued function

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then the (scalar) system of differential equations can be rewritten as one vector differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

Example 1. Let's consider the simple mass-spring system with $k = m$, but this time we'll write it using the variables x and y to be consistent with the `HPGSystemSolver` notation:

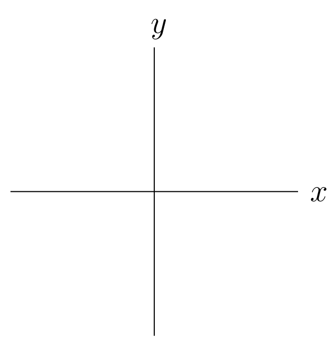
Now let's rewrite this system in vector notation:

Consider one of the solutions that we guessed earlier (translated into the xy-notation):

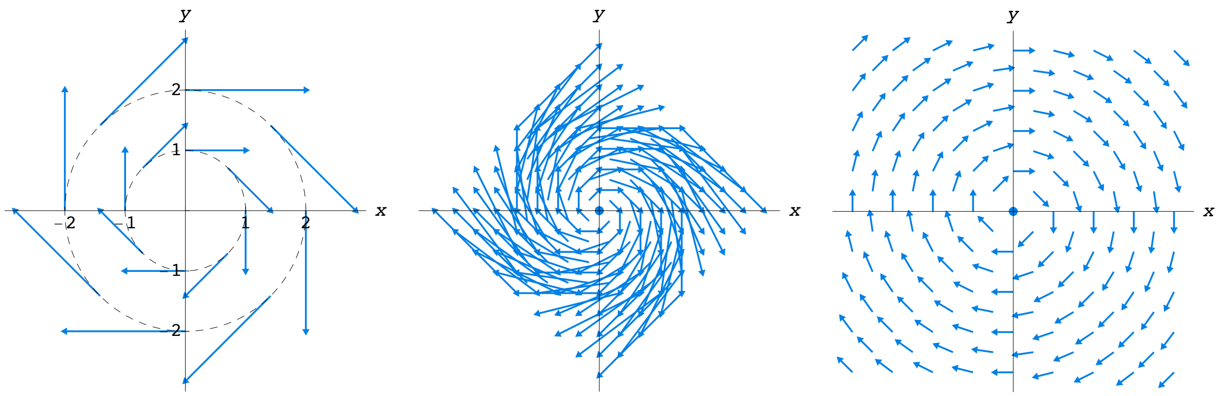
Now for the geometric interpretation of $\dfrac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$, where

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

We use `HPGSystemSolver` to help visualize the vector field and the solutions.



Summary for the simple mass-spring system:

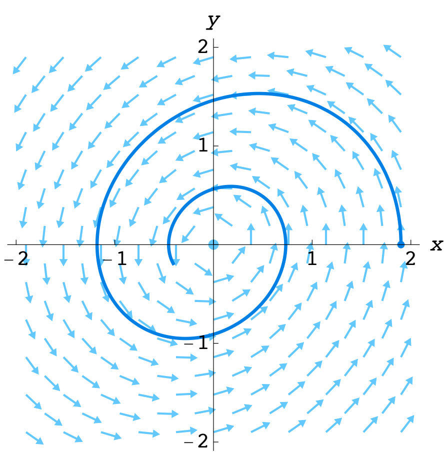


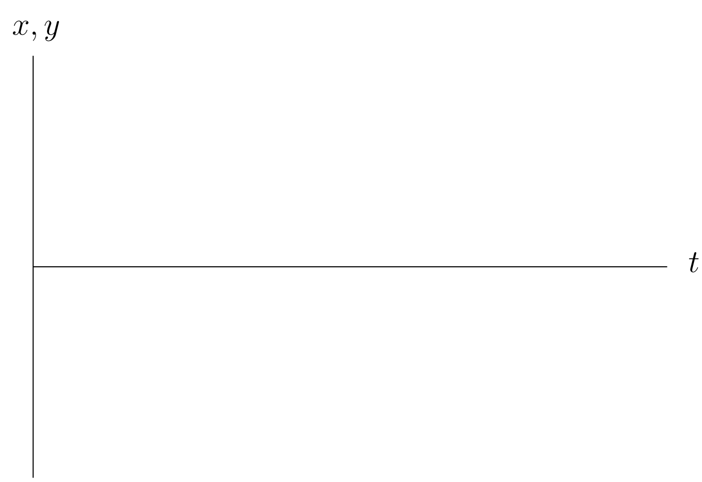
Example 2. Consider the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - 0.3y.$$

The vector field associated with this system is

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x - 0.3y \end{pmatrix}.$$



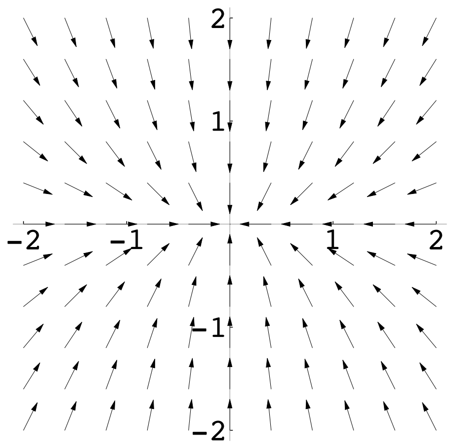


Consider the following 8 first-order systems:

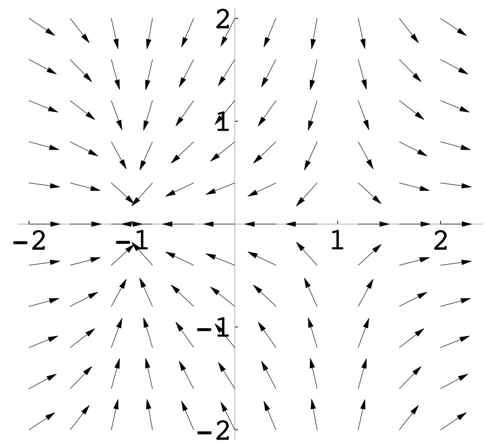
1. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = y^2 - 1$
2. $\dfrac{dx}{dt} = -2x$, $\dfrac{dy}{dt} = -y$
3. $\dfrac{dx}{dt} = -x - 2y$, $\dfrac{dy}{dt} = y$
4. $\dfrac{dx}{dt} = 1 - y$, $\dfrac{dy}{dt} = 1 + x$
5. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = 2x - y$
6. $\dfrac{dx}{dt} = y - 1$, $\dfrac{dy}{dt} = -1 - x$
7. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = -2y$
8. $\dfrac{dx}{dt} = x^2 - 1$, $\dfrac{dy}{dt} = -y$

Four of the associated direction fields are shown below. Pair the direction fields with their associated systems. Provide a brief justification for your choice.

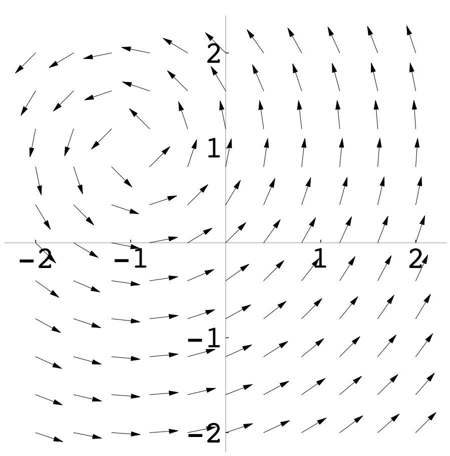
Direction Field A



Direction Field B



Direction Field C



Direction Field D