Forced equations

For the last four weeks of the semester, all of our differential equations have been autonomous. Now we turn to second-order equations that model systems that are subject to some type of external forcing. Here are three examples:

Example. The nonlinear pendulum with a pivot point that is subject to vertical oscillations. The motion of such a pendulum is governed by the second-order nonlinear equation

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\sin\theta = F\sin\theta\cos\omega t$$

where ω determines the frequency of the oscillations of the pivot point and F determines the amplitude of the oscillations. The `Pendulums` tool in `DETools` illustrates this system.

Example. The linear mass-spring system where the spring is subject to vertical oscillations. To model this system, we use the standard mass-spring system and add a term that corresponds to the force added to the system by the oscillations. We get

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F\cos\omega t.$$

The `ForcedMassSpring` tool in `DETools` illustrates this system.

Example. The classic RLC circuit is also modeled by a linear, forced second-order equation. In `DETools`, it is modeled by an equation that involves both charge and current. In our text, we tend to use the equation

$$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = V_s(t)$$

where v_c is the voltage across the capacitor and R, L, and C are the resistance, inductance, and capacitance parameters. The forcing term $V_s(t)$ is a voltage source which can change with time. The `RLCCircuits` tool in `DETools` illustrates this system with a sinusoidal forcing function.

In class we will discuss forced linear equations only, but your second project will involve some experimentation with the forced pendulum.

Our success studying unforced linear systems was due in large part to the Linearity Principle. For forced linear equations, we are fortunate to have the Extended Linearity Principle.

Extended Linearity Principle Consider a nonhomogeneous equation (a forced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

and its associated homogeneous equation (the unforced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

1. Suppose $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the associated homogeneous equation. Then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $k_1y_1(t)+k_2y_2(t)$ is the general solution of the associated homogeneous equation, then

$$k_1y_1(t) + k_2y_2(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

This principle provides the basic framework that we will use to solve linear second-order forced equations. (At this point in the course, you should go back and review the method described in Section 1.8 for solving nonhomogeneous first-order linear equations.)

We already know how to find the general solution to the associated homogeneous equation, so we need only find one solution to the original equation.

Example 1. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{3t}.$$

Here's another example that looks similar but goes somewhat differently.

Example 2. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{-t}.$$

1. The general solution of the associated homogeneous equation is the same as in Example 1.
2. Guessing $y_p(t) = \alpha e^{-t}$ does not produce a solution. Why?
3. If we guess $y_p(t) = \alpha t e^{-t}$, we obtain a solution if $\alpha = -1/3$.

A time saver: There's a calculation that we've already done many times before. It is also useful for guessing $y_p(t)$. Consider the function $y_p(t) = ae^{\lambda t}$ and calculate

$$\begin{aligned} m\frac{d^2y_p}{dt^2} + b\frac{dy_p}{dt} + ky_p &= m(a\lambda^2 e^{\lambda t}) + b(a\lambda e^{\lambda t}) + k(ae^{\lambda t}) \\ &= a\left(m\lambda^2 + b\lambda + k\right)e^{\lambda t}. \end{aligned}$$

Let's see how this works in Example 1.

Example 1. Recall

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{3t}.$$

The characteristic polynomial of this associated homogeneous equation is $p(\lambda) = \lambda^2 - \lambda - 2$. Then we guess $y_p(t) = ae^{3t}$, and we obtain

$$\frac{d^2y_p}{dt^2} - \frac{dy_p}{dt} - 2y_p = (a)(4)e^{3t}$$

because the characteristic polynomial evaluated at 3 is $p(3) = 9 - 3 - 2 = 4$. Therefore, for this nonhomogeneous equation, we want a such that $4a = 1$. We obtain $y_p(t) = \frac{1}{4}e^{3t}$.

Sinusoidal forcing

Now we are going to study forced equations where the forcing function is sinusoidal (either sine or cosine).

Now let's apply this guessing technique to sinusoidally forced linear equations.

Example. Let's calculate the general solution to the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = \cos 2t.$$

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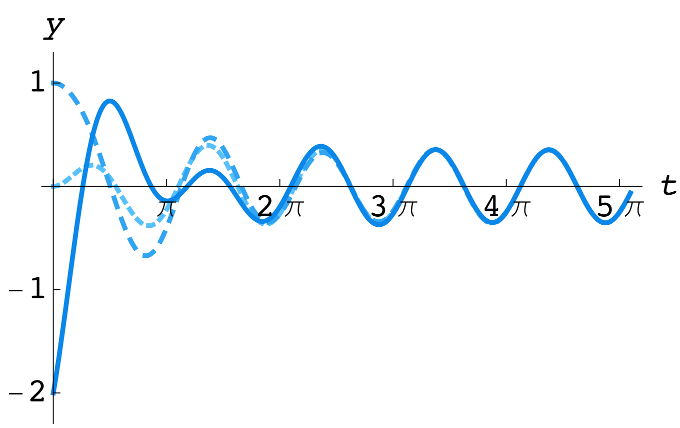
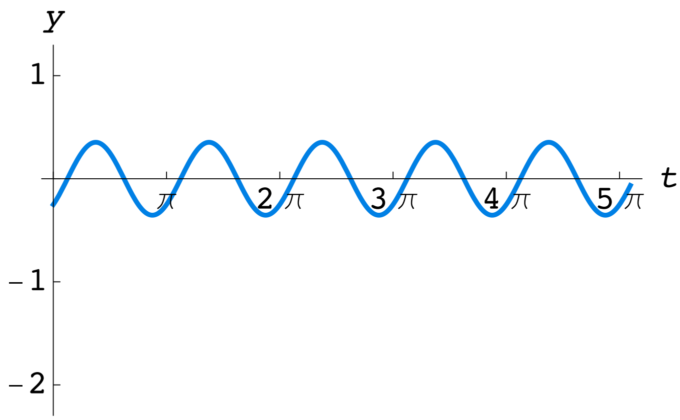
We can see the implications of this computation by entering this equation into `ForcedMassSpring` in `DETools`.

Here are the graphs of three solutions:



Here is the graph of the steady-state solution:



A little translation:

Consider the second-order linear forced equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

where m and k are positive and $b \geq 0$.

Engineering terminology:

forced response—any solution to the forced equation.

steady-state response—behavior of the forced response over the long term.

natural (or free) response—any solution of the associated homogeneous equation.

Why are initial conditions essentially irrelevant?