More on sinusoidal forcing

Last class we considered the following example.

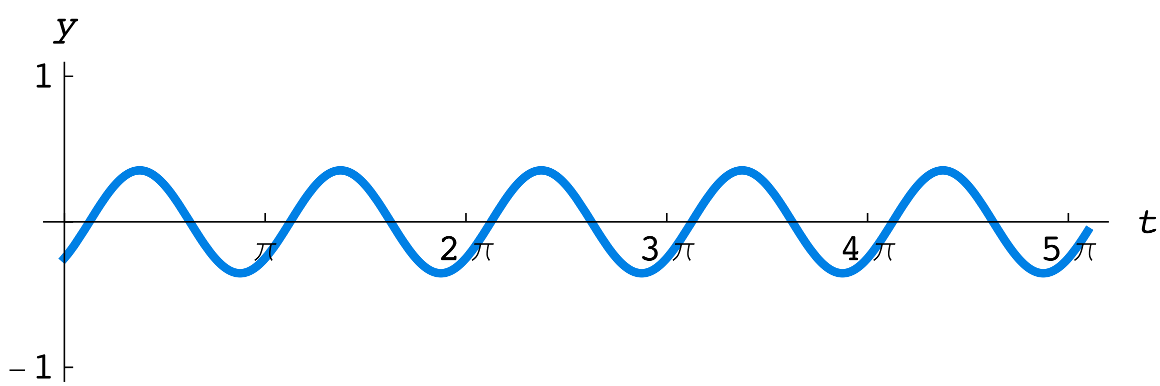
Example. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 2y = \cos 2t$

We calculated the general solution

$$y(t) = k_1 e^{-t/2} \cos\left(\frac{\sqrt{7}}{2}\,t\right) + k_2 e^{-t/2} \sin\left(\frac{\sqrt{7}}{2}\,t\right) - \tfrac{1}{4}\left(\cos 2t - \sin 2t\right).$$

Let's ignore the transient part of this solution and focus on the steady-state solution

$$y_p(t) = -\tfrac{1}{4}\left(\cos 2t - \sin 2t\right).$$



We computed the steady-state $y_p(t)$ using a guessing technique that involved complex numbers. We complexified the differential equation and solved

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = e^{2it}$$

by guessing $y_c(t) = ae^{2it}$. We calculated the complex number

$$a = -\tfrac{1}{4}(1+i),$$

and this number tells us everything we need to know about the steady-state solution.

In order to see why, we use polar coordinates in the complex plane (see pp. 751–755 in Appendix C of the text).

Let's rewrite $a = -\frac{1}{4}(1 + i)$ in this polar form.

What does this polar representation of a tell us about the steady-state solution?

Sinusoidal forcing in the absense of damping

Now consider the mass-spring system without the dashpot.

Example. Let's find the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t.$$

Note the lack of a damping term. We want to see what happens with various forcing frequencies.

Unfortunately the parts of the solution that correspond to the associated homogeneous equation do not die out. So to get some qualitative understanding in this case, we make a simplifying assumption. We consider the solution that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. We obtain

$$y(t) = \frac{1}{3 - \omega^2}\left(\cos \omega t - \cos \sqrt{3}\, t\right).$$

On the web site, there is a Quicktime animation of the graphs of these solutions as we vary the forcing frequency ω. We can also visualize these solutions using a parameter in `HPGSystemSolver`.

The following trig identity helps us interpret what we see in the animation.

Trig identity:

$$\cos at - \cos bt = -2\,(\sin \alpha t)\,(\sin \beta t)$$

where

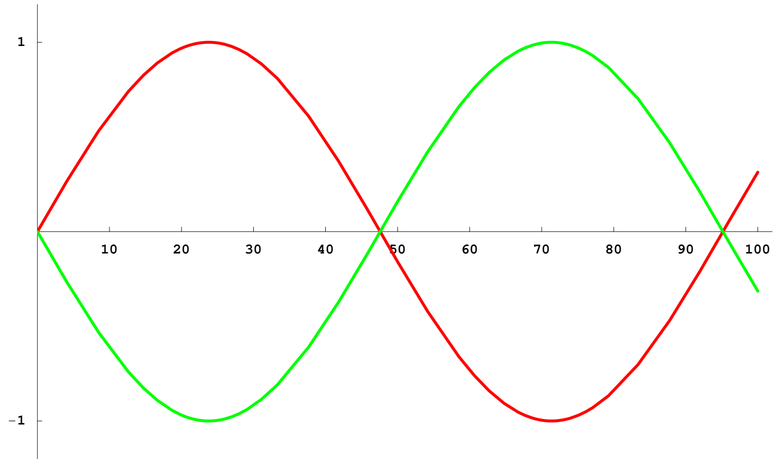
$$\alpha = \frac{a+b}{2} \quad \text{and} \quad \beta = \frac{a-b}{2}.$$

The number α is the average of a and b, and β is called the *half-difference* of a and b.

Example. Let's use this trig identity to get a rough idea of the graph of

$$\cos \omega t - \cos \sqrt{3}\, t$$

where $\omega = 1.6$.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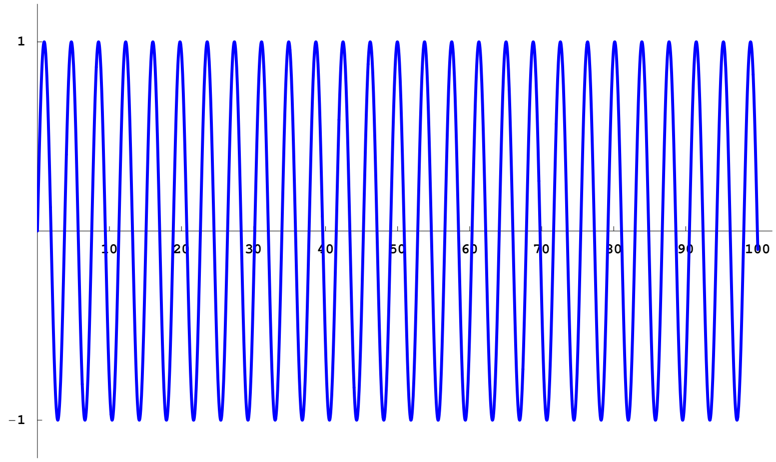
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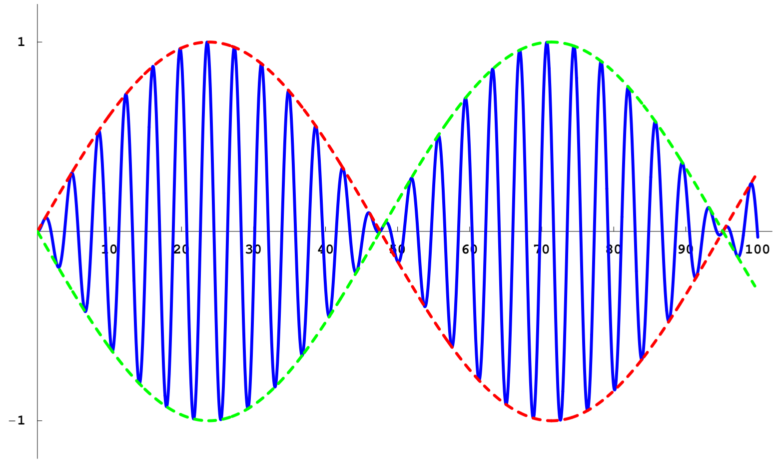
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Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3 - \omega^2}(\cos \omega t - \cos \sqrt{3}\, t).$$

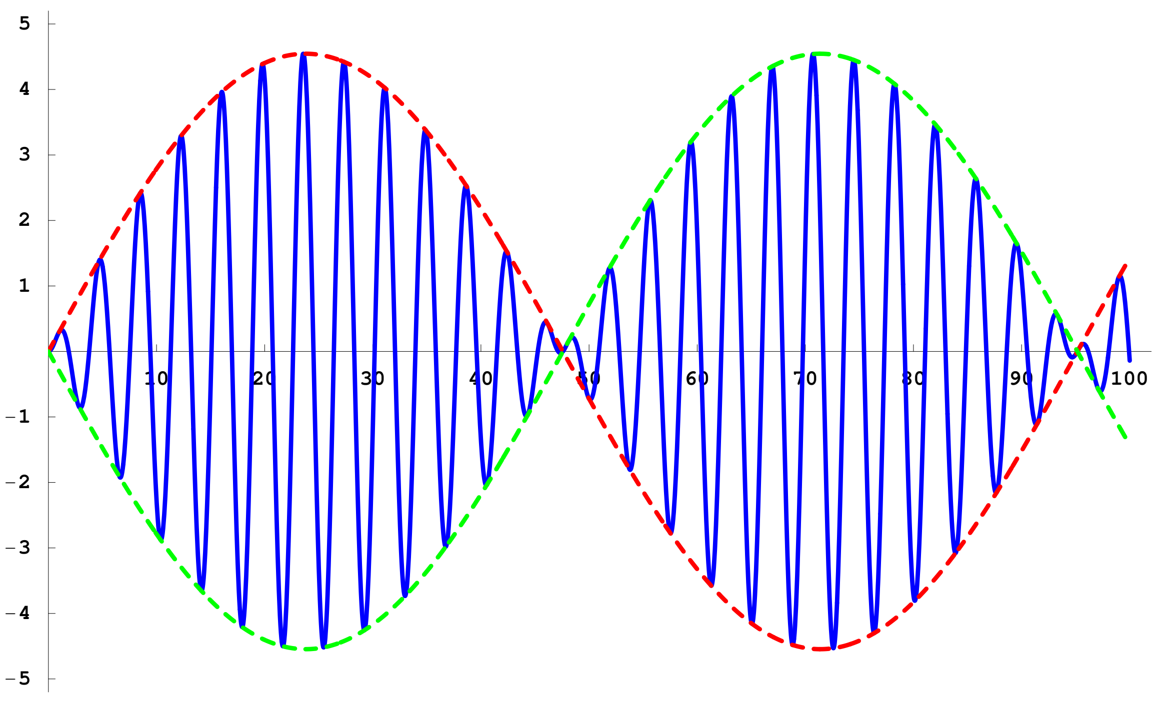
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3 - \omega^2}\ (\sin \alpha t)\,(\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$.



What happens if $\omega = \sqrt{3}$?

Example. $\dfrac{d^2y}{dt^2} + 3y = \cos\sqrt{3}\,t$.

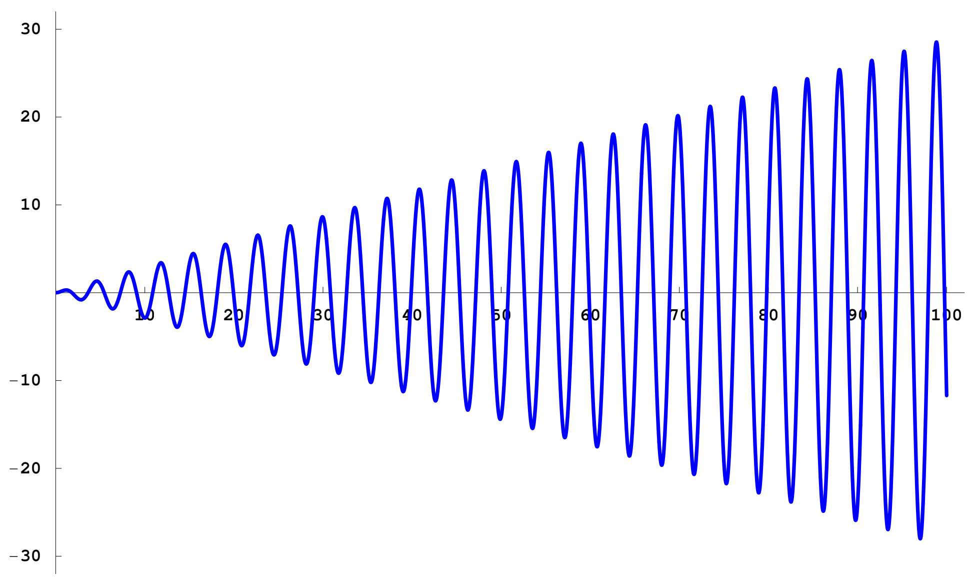
The complexified equation is $\dfrac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}\,t}$. What guess should we use?

Using the guess $y_c(t) = ate^{i\sqrt{3}t}$, we get $a = \dfrac{1}{2i\sqrt{3}} = -\dfrac{1}{2\sqrt{3}}\,i$.

If $\omega = \sqrt{3}$, the general solution is

$$y(t) = k_1\cos\sqrt{3}\,t + k_2\sin\sqrt{3}\,t + \frac{1}{2\sqrt{3}}\,t\,\sin\sqrt{3}\,t.$$

Here is the graph for the case where $k_1 = k_2 = 0$.



This value of ω is called the resonant value for the frequency of the forcing.

The resonance value of the forcing should be immediately apparent from the differential equation.

Example. What is the resonance value of ω for the one-parameter family of equations

$$\frac{d^2y}{dt^2} + 5y = 4\cos\omega t\,?$$