Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 3)^2$, so there is only one eigenvalue, $\lambda = 3$. Let's calculate the associated eigenvectors:

But we already know how to solve this system. How?

We obtain the general solution

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} + 2y_0 t e^{3t} \\ y_0 e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t} \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}.$$

Note that this general solution is *not* written as a linear combination. Every nontrivial solution contains the first term, and most solutions contain both terms.

We use this result to motivate a different technique that we use to solve systems with repeated eigenvalues. We guess a solution of the form

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + t e^{\lambda t}\mathbf{V}_1.$$

Note that the initial condition for this solution is $\mathbf{V}_0$.

Fact from linear algebra: If $\mathbf{A}$ is a 2×2 matrix with a repeated eigenvalue λ and $\mathbf{V}_0$ is any vector, then either

1. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0 = \mathbf{0}$ (in other words, $\mathbf{V}_0$ is an eigenvector), or
2. the vector $\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0$ is an eigenvector of $\mathbf{A}$.

Example. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

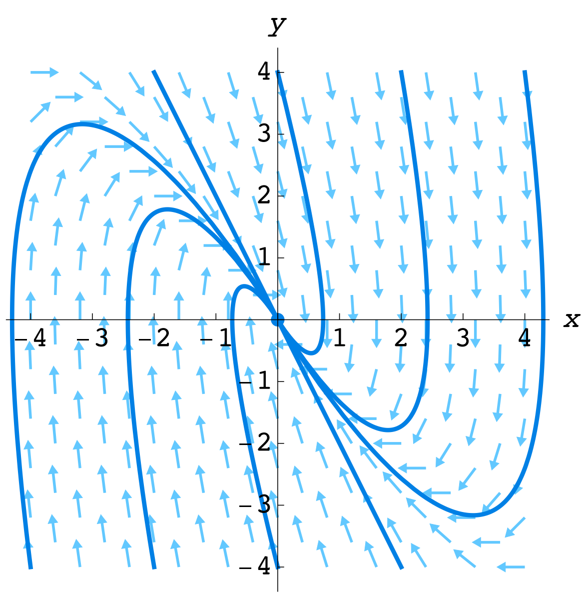
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$

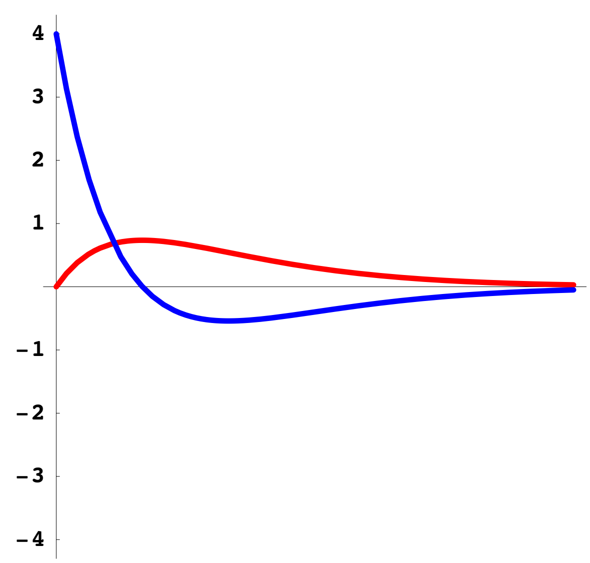
The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 4\lambda + 4$, so $\lambda = -2$ is a repeated eigenvalue.

What is the long-term behavior of a system with a repeated, negative eigenvalue?

It is interesting to look at this example using two of the tools on the CD. Using `LinearPhasePortraits`, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use `HPGSystemSolver` to plot the phase portrait and a typical pair of $x(t)$- and $y(t)$-graphs.



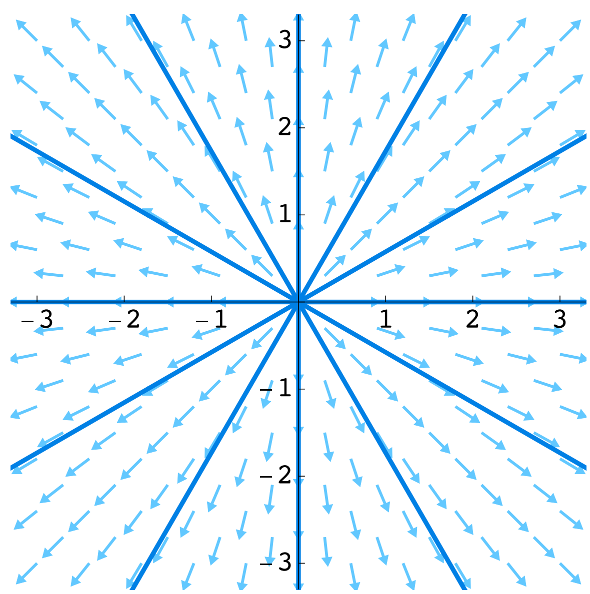


Unusual case of repeated eigenvalues: There is one type of linear system that has repeated eigenvalues that is different from the examples we have discussed.

Example. Consider $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ where $\mathbf{A}$ is the diagonal matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

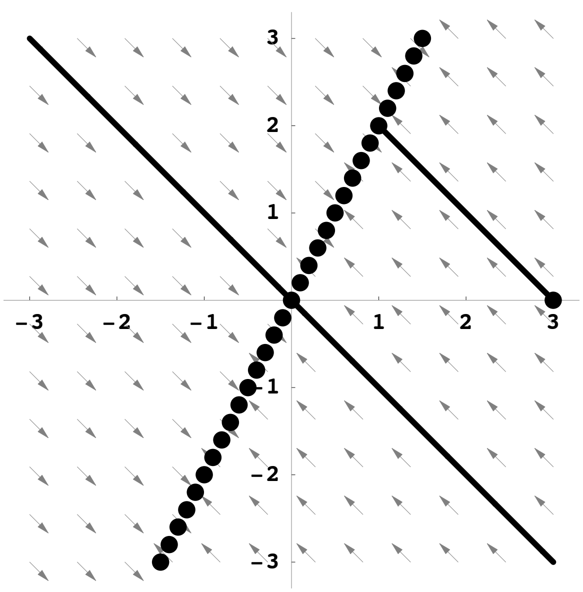
What are its eigenvalues and eigenvectors?

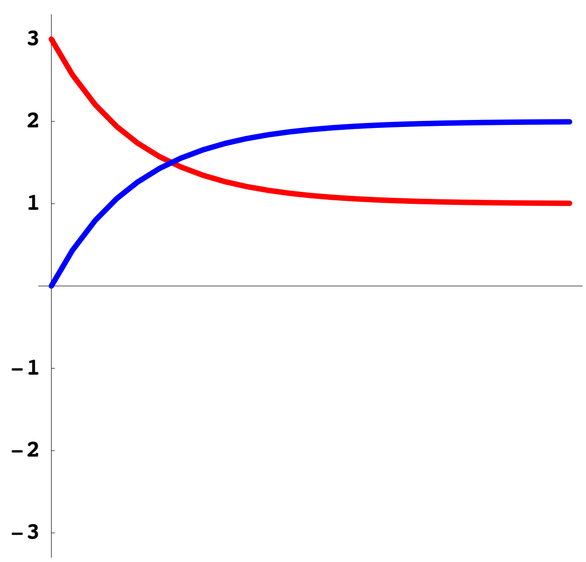


Finally consider the example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}.$$

Its characteristic polynomial is $\lambda^2 + 3\lambda$. So its eigenvalues are $\lambda = -3$ and $\lambda = 0$. (If a system has 0 as an eigenvalue, we say that it is *degenerate*. The matrix $\mathbf{A}$ of coefficients is singular—see your class notes for March 24.)





Second-order, linear equations

We now apply what we have learned about linear systems to solve second-order homogeneous linear equations.

Let's return to the guessing technique for second-order equations that we learned about a month ago (see Section 2.3 in the text and your class notes from March 17). In particular, let's see how it relates to what we have done with linear systems recently.

Example. Consider the equation $2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 4y = 0.$

1. Use a guessing technique to find two nonzero solutions $y_1(t)$ and $y_2(t)$ that are not multiples of each other.

2. Convert this equation to a first-order system and determine the analogous solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

3. In what way are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ special solutions?

Let's see how this guessing technique can be used to solve all second-order homogeneous equations.

Consider

$$a\,\frac{d^2y}{dt^2} + b\,\frac{dy}{dt} + cy = 0$$

with its characteristic equation $a\lambda^2 + b\lambda + c = 0$ as well as the corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}\,y - \frac{b}{a}\,v \end{aligned}$$

with its characteristic equation

$$\det\begin{pmatrix} -\lambda & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If λ is an eigenvalue, the vector $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ is *always* an associated eigenvector.

Let's see what that observation tells us about solutions to the second-order equation. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a homogeneous linear second-order equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a linear second-order equation.