More on beats and resonance

Last class we discussed the solutions to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfy the initial condition $(y(0), y'(0)) = (0, 0)$ where ω is a parameter. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3 - \omega^2}(\cos \omega t - \cos \sqrt{3}\, t).$$

Applying a trig identity, we obtain

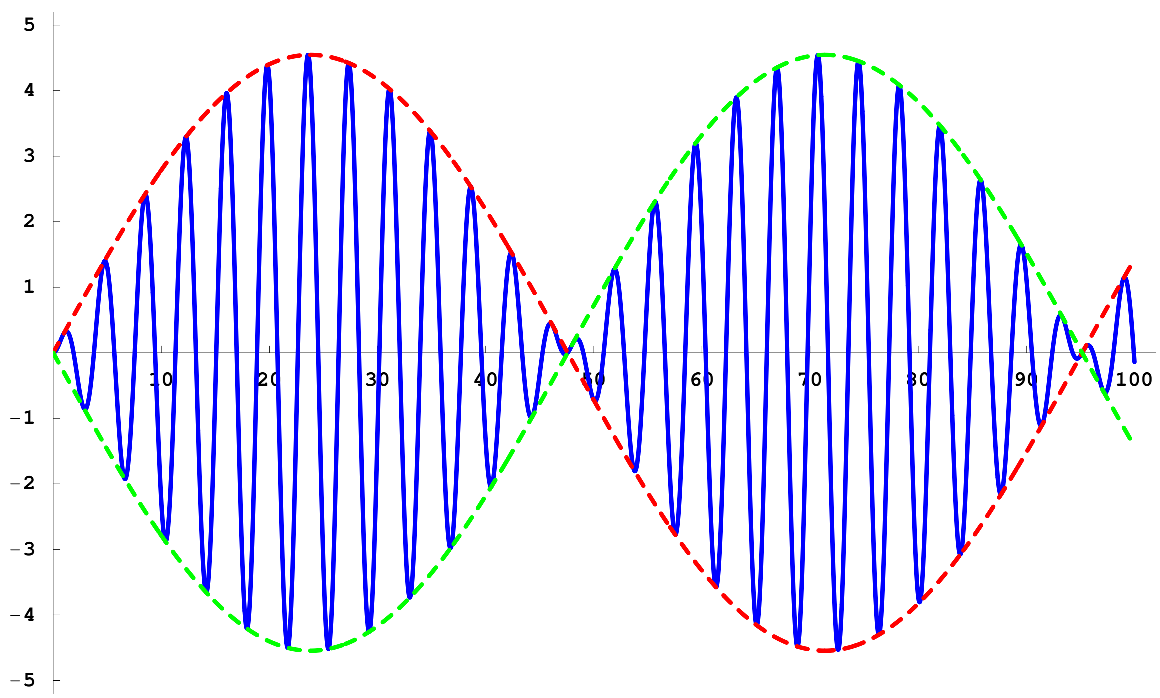
$$y(t) = \frac{-2}{3 - \omega^2}\; (\sin \alpha t)\, (\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$. Note that the average of ω and $\sqrt{3}$ in this case is approximately 1.67. The half difference is approximately -0.066. The average yields "rapid" oscillations with a period of approximately 3.76. The half-difference yields "slow" oscillations with a period of approximately 95. Also,

$$\frac{2}{3 - 1.6^2} \approx 4.55.$$



What happens if $\omega = \sqrt{3}$?

Example. $\dfrac{d^2y}{dt^2} + 3y = \cos\sqrt{3}\,t$.

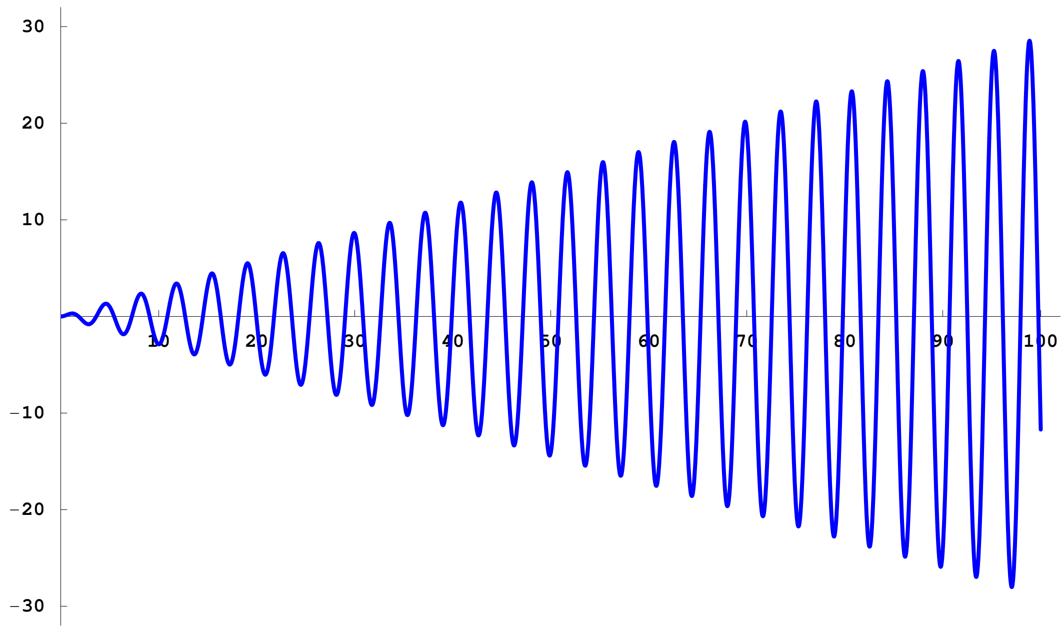
The complexified equation is $\dfrac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}\,t}$. We guess $y_c(t) = ate^{i\sqrt{3}t}$, and we get

$$a = \frac{1}{2i\sqrt{3}} = -\frac{1}{2\sqrt{3}}\,i.$$

Consequently, if $\omega = \sqrt{3}$, the general solution is

$$y(t) = k_1 \cos\sqrt{3}\,t + k_2 \sin\sqrt{3}\,t + \frac{1}{2\sqrt{3}}\,t\,\sin\sqrt{3}\,t.$$

Here is the graph for the case where $k_1 = k_2 = 0$.



This value of ω is called the resonant value for the frequency of the forcing.

The resonance value of the forcing should be immediately apparent from the differential equation.

Example. What is the resonance value of ω for the one-parameter family of equations

$$\frac{d^2y}{dt^2} + 5y = 4\cos\omega t\,?$$

Linearization

We would like to apply what we know about linear systems to nonlinear systems.

Example. Consider the van der Pol equation

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.$$

The corresponding system is

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = (1 - x^2)y - x.$$

The only equilibrium point for this system is $(0, 0)$. What is the linearized system near $(0, 0)$?

Example. Consider the (undamped) pendulum

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

The corresponding system is

$$\frac{d\theta}{dt} = v$$

$$\frac{dv}{dt} = -\sin\theta.$$

There are equilibria at $(\theta, v) = (k\pi, 0)$ for all integers k.

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{Y}.$$

What is the linearized pendulum near the equilibrium point $(\pi, 0)$?

Given the (nonlinear) system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

its **Jacobian** at the point (x_0, y_0) is the matrix

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

and its linearization at (x_0, y_0) is the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}.$$

For the pendulum, we have one linearization for each equilibrium point:

For the van der Pol system, we obtain the linearization:

Linearization Theorem Let $\mathbf{Y}_0$ be an equilibrium point for the nonlinear autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

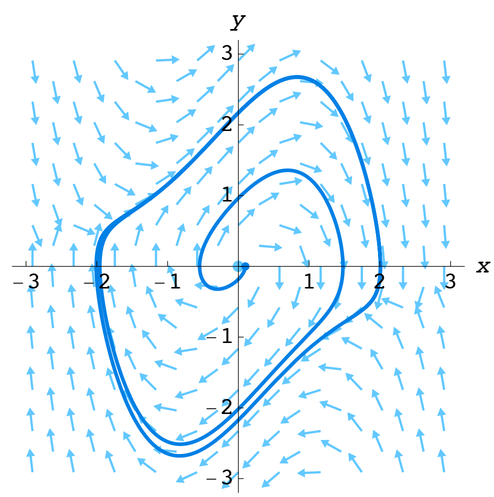
and let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}$$

be the corresponding linearized system. If $\det \mathbf{J} \neq 0$ and the eigenvalues of $\mathbf{J}$ are not purely imaginary, then the solution curves of the nonlinear system near $\mathbf{Y}_0$ behave in the same qualitative way as the solution curves of the linear system.

Example. Consider the van der Pol system near the origin. The linearized system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$



Example. Consider the pendulum system. The linearized system near $(\pi, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

Its characteristic equation is $\lambda^2 - 1 = 0$, and therefore the eigenvalues are ± 1. The Linearization Theorem says that this equilibrium point is a nonlinear saddle.

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y},$$

and the characteristic equation is $\lambda^2 + 1 = 0$. The eigenvalues are $\pm i$. This equilibrium point is a nonlinear center, but this example is misleading. The Linearization Theorem does not apply to this equilibrium point.